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Improving Cooperation in Public Goods Game through Fractional Punishment of Free-Riders

Magdalena del Rocío Botta Solano López

Tesis presentada a la Facultad Politécnica, Universidad Nacional de Asunción, como requisito para la obtención del Grado de Doctor en Ciencias de la Computación.

San Lorenzo - Paraguay

Octubre, 2021

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Magdalena del Rocío Botta Solano López

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A los que vinieron antes. Sus historias son las ramas que sostienen y dan forma, muchas veces sin darme cuenta, a la mía.

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“He conocido lo que ignoran los griegos: la incertidumbre”
(Jorge Luis Borges, La lotería en Babilonia).

Mejorando la Cooperación en el Juego de Bienes Públicos a través del Castigo Fraccionado de los Infractores

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Resumen

En la mayoría de las iniciativas conformadas por un grupo de individuos unidos para lograr un objetivo común, aquellas personas que se benefician del grupo sin contribuir a crear y sostener la iniciativa (denominados desertores), son causa de los malos resultados del proyecto. Este comportamiento, cuando pasa desapercibido, se propaga fácilmente dentro del grupo. Desde un punto de vista personal y pensando a corto plazo, es beneficioso ser un desertor, sin embargo, si todos desertan, la iniciativa fracasa. Mejorar la cooperación o, dicho de otra forma, reducir el número de desertores es clave en el éxito de un proyecto. Con este fin, recompensar a quienes siguen las reglas o sancionar a los desertores son las medidas más comúnmente aplicadas. Sin embargo, crear y sostener un sistema de incentivos es costoso. Consume recursos tales como personal, dinero y tiempo y, como consecuencia, reduce las ganancias del grupo. A veces, incluso con los recursos necesarios, es logísticamente imposible castigar a todos los desertores. En este sentido, es una práctica común, tanto en las iniciativas públicas como privadas, sancionar solo a una fracción de los infractores. Incluir este conocimiento empírico en un modelo matemático de evolución de la cooperación mediante un juego de bienes público con la dinámica del replicador, es la propuesta de esta tesis.

Palabras clave: Castigo fraccionado, juego de bienes públicos opcional, evolución de la cooperación, teoría de juegos evolutivos, dinámica del replicador.

Improving Cooperation in Public Goods Game through Fractional Punishment of Free-Riders

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Summary

In most human initiatives involving non-related people joined together to achieve a common goal, free-riders cause poor results. They profit from the group without contributing to create and sustain the initiative. This behavior, when left unnoticed, often spread within the group. It is beneficial to be a free-rider; however, if everyone follows the same strategy, the initiative fails. To improve cooperation or, conversely, to reduce free-riding is key to successful enterprises. To this end, rewarding those who follow the rules or sanctioning free-riders are commonly used methods. However, a system of incentives is expensive. It consumes resources such as people, time, and money, consequently reducing the group's profit. Sometimes, even with the resources, it is logistically impossible to penalize every free-rider in the group. Therefore, it is common to sanction only a fraction of the free-riders in public and private initiatives in this vein. This thesis proposes to include this empirical knowledge into an evolution of cooperation mathematical model through an optional public goods game with replicator dynamics.

Keywords: Fractional punishment, optional public goods game, evolution of cooperation, evolutionary game theory, replicator dynamics.

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Glossary

J Jacobian matrix.

V Lyapunov function.

$\bar{p}$ population average payoff.

$\hat{f}$ fraction of cooperators among players at the equilibrium.

$\hat{f}_0$ fraction of cooperators among players at the equilibrium for $d = 0$.

$\hat{f}_d$ fraction of cooperators among players at the equilibrium for $d > 0$.

$\hat{x}$ frequency of cooperators at the equilibrium.

$\hat{y}$ frequency of defectors at the equilibrium.

$\hat{z}$ frequency of loners at the equilibrium.

$\hat{z}_0$ frequency of loners at the equilibrium for $d = 0$.

$\hat{z}_d$ frequency of loners at the equilibrium for $d > 0$.

$\mathcal{S}_3$ simplex.

σ loner's payoff.

c common pool contribution.

f fraction of cooperators among players.

n number of individuals in the sample.

n_x number of cooperators in the sample.

n_y number of defectors in the sample.

n_z number of loners in the sample.

p_x cooperator's payoff.

p_y defector's payoff.

p_z loner's payoff.

r multiplication factor.

s number of defectors and cooperators in the sample.

x frequency of cooperators.

y frequency of defectors.

z frequency of loners.

CPR Common Pool Resources.

EGT Evolutionary Game Theory.

ESS Evolutionarily Stable Strategy.

JS Juntas de Saneamiento / Sanitation Boards.

SB Sanitation Boards / Juntas de Saneamiento.

SDG Sustainable Development Goals.

Chapter 1

Introduction

This thesis presents the fractional sanction or punishment model as one possible mechanism that increases cooperative behavior in a group. The proposal frames the model in the context of evolutionary game theory and, particularly, based on models that study the appearance and maintenance of cooperation using public goods games.

Although not explicitly described or modeled, the inspiration comes from ordinary practices in institutions, community projects, and, organized groups where free-riding, an uncooperative behavior, becomes a problem. Free-riding affects the performance and survival of a project often requiring to penalize those who break the rules. However, sanctioning implies a cost for the project; the group may decide between sanctioning the free-riders or assuming the financial loss due to resource utilization (personnel, time, and money). There is no single recipe for how to punish free-riders. Different groups find different solutions. One of them is to sanction only a small group of them; cost reduction benefits this method. An example of this methodology is random inspections. This strategy seeks to dissuade the remaining population from breaking the norms.

Using evolutionary game theory and evolutionary models that analyze cooperation (and, consequently, free-riding) provides a different perspective of the problem. For example, in a project with a high rate of non-payment, where the resources are not enough to sustain the system, it may be sensible to raise the service fee or the individual contribution to cover the expenses. Increasing the contribution may accomplish the purpose of maintaining the service, but, this situation is counterproductive in the long term analyzed from the evolution of cooperative behavior within a population point of view. Increasing the contribution is, in practice, sanctioning those who do comply with the rules. In addition, this measure inadvertently stimulates the increase of free-riders in the population because the benefit for those who contribute decrease.

In the literature, the prisoner's dilemma exemplifies situations in which two individuals must decide whether to cooperate or not. Not knowing, each of them, how the other will act, the rational option is not to cooperate, and the result is that both end up in a worse situation than they would have been if they had cooperated. If more than two individuals participate in the dilemma, the literature defines it as a public goods game. In this game, as in the prisoner's dilemma, the rational choice, not cooperating, puts the group in a worse situation than cooperating. When in the game, individuals can abstain from playing, this situation becomes an optional public goods game. In this work, the results for both cases are analyzed, emphasizing the optional public goods game that presents a greater diversity of situations and outcomes.

The evolution of cooperation study, emphasizes the mechanisms that favor the appearance and maintenance of cooperation in a population. Undoubtedly, the best known and most applied mechanism is to use incentives to increase cooperation, whether positive (reward) or negative (sanction). Another well-studied mechanism is the individual's possibility to abstain from participating in a game or project. This mechanism, rather than increasing cooperation, facilitates its maintenance over time.

From the dynamics used in public goods games to study the evolution of cooperation within a population, the best known is the replicator dynamics. According to this dynamic, a behavior increases its frequency in a population if the individual strategy benefit exceeds the average population benefit. On the contrary, if the benefit of the behavior is less than the average benefit, its frequency decreases, and it tends to disappear. The intuition behind this process can relate to people's tendency to constantly compare their behavior and the benefit they obtain from it with others' behavior and benefit, to imitate the one providing the most significant benefit.

The fractional sanction model presented in this thesis is theoretical. However, the system's behavior is similar to that observed empirically in real cases such as institutions and groups of individuals working on a joint project. Much of this work was motivated by the community's drinking water supply system, Paraguay's Sanitation Boards (SBs). The fundamental problem for these boards to subsist is free-riding. Over time, many present cyclical behaviors of good initial performance, followed by a decrease in contributions, or in other words, a reduction of cooperation that leads to poor performance and the near disappearance of the system. This cyclical behavior is very similar to the results observed in an optional public goods game. In adition, SBs have problems sanctioning free-riders, often facing both resource and social constraints to sanction all those who break the rules.

The proposal of this thesis is modeling a fractional punishment in public goods games. Hence, previous studies were carried out and are included herein in chronological order. Therefore, the work organization is as follows: Section 1.1 presents the research objectives. Section 1.2 summarizes publications and participation in international conferences in relation to this thesis. the bibliographic review is in Chapter 2. The research area is highly interdisciplinary; therefore, the chapter briefly reviews the origin and main components. Likewise, the review presents the critical works in the area and the most recent works that served as the basis for this thesis. Chapter 3 portrays, facilitating reading and not duplicate content, the concepts, and methodology that are common to the following chapters.

Chapter 4 presents the results of the first part of the research that constitute the background of this work. Chapter 5 defines the fractional sanction model for two strategies; this, is a compulsory public goods game. The work focuses on of free-riding in drinking water supply systems in sanitation boards and was presented at the CCIS (Botta et al., 2020) conference. Later, Chapter 6 presents the sanction model for an optional public goods game (with three strategies).

Finally, chapters 7, and 8 respectively summarize the discussion of the results and findings, and conclusions of the thesis. After the Bibliography, an extended resume in spanish is presented.

1.1 Research objective

- To define an evolutionary game theory model of the fractional punishment for public goods games, seeking mechanism to increase cooperation and the requirements to achieve full cooperation, in consistency with empirical observations on practical cases.

1.2 Specific objectives

- To define the fractional punishment mechanism.
- To model the system in the context of evolutionary game theory and the evolution of cooperation.
- To present the fractional sanction for optional and compulsive public goods games using replicator dynamics.
- To analyze the results obtained concerning the modification of the equilibrium points and the cooperators' frequency growth.

- To describe the conditions to achieve full cooperation.
- To point out the similarities between the theoretical results and empirical observations in practical cases.

1.3 Publications and international conferences

Publications:

- Botta, R., Blanco, G., y Schaerer, C. E. Fractional punishment of free riders to improve cooperation in optional public good games. Games, 12(1), (2021).
- Botta, R., Blanco, G., y Schaerer, C. E. Evolution of cooperation in evolutionary games for Sanitation Boards. CLEI Electronic Journal, 17(2):1–17, (2014).

International Conferences:

- Botta, R., Schaerer, C. E., y Blanco, G. Fractional punishment with redistribution in a compulsory public good game. In: 15th International Conference on Game Theory and Management, GTM. 2021. Russia (Online event).
- Botta, R., Schaerer, C. E., y Blanco, G. Cooperation and punishment in community managed water supply system. In: Conference of Computational Interdisciplinary Science, CCIS. 2019. EUA.
- Botta, R., Blanco, G. y Schaerer, C. E. Control of incentives for promoting cooperation in sanitation boards. In: 13th European (formerly Spain-Italy-Netherlands) Meeting on Game Theory, SING13. 2017. France.
- Botta, R., Blanco, G., y Schaerer C. E. Applying evolutionary games to sanitation boards. In: XXXVII Congresso Nacional de Matemática Aplicada Computacional, CNMAC. 2017. Brazil.
- Botta, R., Blanco, G. y Schaerer, C. E. Finite population evolutionary game for water supply sanitation boards. In: Conference on Mathematical Modeling and Control of Communicable Diseases, MMCCD. 2016. Brazil.

Chapter 2

Literature review

Cooperation, as a behavior, is difficult to explain. Examples in which individuals of the same species cooperate with each other abound in nature, yet the very existence of cooperation defies logic. Natural selection is a competitive process that favors the fittest individuals, that is, individuals with traits and behaviors that favor their survival. Eventually, by inheritance, those favorable characteristics will pass down to the next generation. However, a cooperative individual gives up a part of his fitness to benefit others, under these conditions, how is it possible that cooperation exists and survives?

For decades, this riddle has puzzled evolutionary biologists studying altruism (Hamilton, 1964a; Haldane, 1955). The cooperation existence can be explained under kin selection (Hamilton, 1964b), direct reciprocity (Trivers, 1971; Axelrod and Hamilton, 1981), indirect reciprocity (Nowak and Sigmund, 1998) ,and structured population (Nowak and May, 1992). In each case, there are conditions for cooperation to succeed, for instance, direct reciprocity requires a small group of participants and repetitive interactions; kinship is based on a genetic relationship. Indirect reciprocity requires knowing the reputation of all participants, and structured population requires that each participant is positioned such that the interacting participants are always the same (Nowak, 2006). When the conditions mentioned above are not fulfilled, for instance, in large populations, without spatial population structure or no-kinship within the group, the individuals select, free-riding.

But, this enigma is not restricted to natural sciences, cooperation is relevant for contrasting areas of knowledge. In social sciences, this result receives different denominations such as social dilemma (Dawes, 1980), the tragedy of the commons (Hardin, 1968, 1998), free-rider problem, and social trap (Hauert et al., 2002b). A social dilemma describes a situation where a defecting behavior as free-riding is more beneficial than cooperating, independently of what other individuals choose to do. Still, if all cooperate, the benefit is greater than if all defect (Dawes, 1980).

Likewise, in social sciences, proposals were made to understand and improve cooperation, in “The Tragedy of the Commons” (Hardin, 1968), Hardin points out how overexploitation of shared resources for personal benefit can lead to resource depletion that eventually is detrimental for all. To improve cooperation, Hardin proposes “Mutual Coercion Mutually Agreed Upon”. In a complementary insight, Ostrom, in “Governing the commons” (Ostrom, 1990), enumerates principles observed in successfully managed Common Pool Resources (CPR).

Natural and social sciences, analyzed the situation using game theory (Trivers, 1971; Axelrod and Hamilton, 1981). In 1971, (Trivers, 1971) related cooperation with game theory when suggesting that the association between two individuals exposed repeatedly to symmetric and reciprocal situations, is analogous to the Prisoner's Dilemma. Afterward, in “The Evolution of Cooperation” (Axelrod and Hamilton, 1981), Axelrod models direct reciprocity through iterated prisoner's dilemma, by two computer tournaments organized to find a strategy able to sustain cooperation. In both tournaments the winner strategy was tit-for-tat.

The development of the evolutionary game theory (EGT) was a relevant event for studying the evolution of cooperation. In 1973, J. Maynard Smith and G. R. Price published “The Logic of Animal Contest" (Smith and Price, 1973) to explain the logic behind the ritualized contest among animals. EGT incorporates elements from classical game theory (Von Neumann and Morgenstern, 1953) and natural selection theory (Darwin, 1859).

Unlike classic game theory, EGT, is not focused on rational players making decisions. EGT deals with populations of individuals, and it is the natural selection process that determines the evolution of a strategy (trait or behavior) in a population. To apply game theory in a biological context of conflict and cooperation, it is necessary to replace “utility" by “fitness" and “rationality" by “natural selection" (Smith, 1986). Strategies with higher benefits (payoffs) are favored by natural selection and spread within the population by imitation, learning or inheritance (Hofbauer and Sigmund, 2003).

A strategy's payoff depends on the action taken by others; this is, it depends on the frequencies of the strategies present in the population. Since these frequencies change under the payoffs, this leads to a feedback loop. (Hofbauer and Sigmund, 2003).

In (Smith and Price, 1973), the concept of an evolutionarily stable strategy (ESS) which can not be invaded when adopted by all in the population, was introduced. However, the ESS was a static concept (Zeeman, 1980); a dynamic for the game was not defined until a few years later. In 1978, Taylor and Jonker (Taylor and Jonker, 1978) proposed a dynamic where the frequency of a strategy increases or decreases according to how well it pays compared to the average payoff in the population. This dynamic was called, later on, “replicator dynamics”.

When dynamical settings were included in EGT, dynamical systems methods were applied to analyze the evolution of cooperation in a population. It was possible to analyze the stability of the equilibrium points in the system and how the strategies frequencies develop in time to converge to the stable states in a population. (Zeeman, 1980; Weibull, 1997; Hofbauer and Sigmund, 1998; Sandholm and Ansell, 2010)

In addition to explaining how cooperative behavior appears, it is relevant to understand how cooperation can be improved or, at least, sustained in time. Two mechanisms that affect the cooperation were found out: incentives and abstention. Rewarding cooperators (positive incentives) or punishing free-riders (negative incentives) can improve cooperation in a group (Yamagishi, 1986; Kollock, 1998; Fehr and Gächter, 2002; Sigmund, 2007; Sasaki et al., 2012), while abstention to participate in a project is useful to sustain cooperative behavior (Hauert et al., 2002a,b).

However, an incentive system is expensive and, therefore, raises a second-order social dilemma (Hilbe and Sigmund, 2010); for this reason, an incentive system is a public good itself (Yamagishi, 1986; Sigmund, 2007).

In public goods games, the two best-studied forms of punishment are: peer punishment and pool punishment (Sigmund et al., 2010b). When peer punishment is applied, every cooperator that decides to punish defectors is called a punisher; after the game, every punisher reduces its benefit by punishing every defector in the population. If pool punishment is applied, before the game, each cooperator contributes to a pool that will be later used to punish the defectors; this pool is separated from the cost the player has to pay to participate in the game. Both methods have some disadvantages.

Cooperators who do not punish are free riders in the sanctioning system (called second-order free-riders). With peer punishment, the cost for a small number of punishers in a group with mostly defectors, prevents the punishers from prosper. Conversely, in a group without defectors, cooperators and punishers have the same payoff; therefore, punishers tend to disappear, allowing defectors to invade the population. This can not occur with pool punishment since the cooperators must contribute in a pool for punishment before the game. Pool punishment however is a fixed cost, that exists even when there are no defectors in the population (Traulsen et al., 2012; Sigmund et al., 2010b).

Both sanctioning methods were studied and compared in theoretical models and experiments (Fehr and Gächter, 2002; Sigmund et al., 2010a,b; Hilbe et al., 2014). For instance, it was found that pool punishment can stabilize cooperation if second-order free-riders are also punished (Sigmund et al., 2010a; Traulsen et al., 2012). In addition, the second-order free-riding problem was analyzed in (Ozono et al., 2017; Hilbe et al., 2014; Sasaki and Unemi, 2011).

Peer and pool punishment are rarely applied in their pure form in practical situations. More recently, some authors have explored several ways to apply punishment, often inspired in real-world situations. When peer punishment is applied, the intrinsic over punishing was treated in (Dercole et al., 2013). In this model, a defector faces a fixed amount of punishment which costs is shared between the punishers; this is often a milder punishment regarding peer punishment where the punishment imposed depends on the number of punishers in the group. In (Chen et al., 2014), a fraction of the cooperators are randomly selected to perform as punishers. They will equally share the cost associated with punishing the defectors in the population. Given that punishment is costly, cooperators avoid punishing defectors; with this mechanism, the cost is fairly distributed. To promote cooperation and regulate the outcome in a public goods game, (Zhang et al., 2017b, 2018) introduce ceiling payoff for defectors. If the number of cooperators surpasses a defined threshold then the payoff for a defector would not increase further.

This work assumes that both, the service and the sanctions must be implemented with the fee collected periodically in a fee-pool. This is observed as a common practice in institutions that provide a service. Within this framework, second-order free-riding does not occur since those who pay for the service implicitly pay for the sanctioning system. However, this also implies that each player's profit is reduced, given that a part of the collected resources must be used to penalize free-riders.

The institution decides the fee-pool portion used for each expense: providing the service and sanctioning free-riders. The sanctioning system may not be a fixed cost; when free-riding decreases, the fee-pool amount required to sanction free-riders will also decrease. Besides, the institution may limit sanctioning expenses by penalizing only a fraction of them.

Chapter 3

Methodology

Consider a large population where from time to time n randomly selected individuals are offered to play in a public goods game ($n = n_x + n_y + n_z$; $n \geq 2$). Those who refuse to play are known in the game as loners (n_z). The loners have a payoff σ that is constant and independent of the game. Those who accept to play the public goods game will receive their payoff from the game ($s = n_x + n_y$). Each player must decide if either contribute or not with a positive amount c to the common pool in the game. Individuals who contribute are cooperators (n_x), those who do not are defectors (n_y). The total contribution to the common pool (obtained from cooperators) is then multiplied by a factor r ($1 < r < n$) and distributed between all the players equally, regardless if they are cooperators or defectors. However, in this model, a previously defined fraction of the defectors will reduce their payoff reduce to zero, while the remaining defectors will obtain their usual payoff.

As defined above, there are three possible strategies; therefore, a player can be a cooperator x, a defector y or a loner z. The payoff of each strategy in a group of n_x cooperators, n_y defectors, and n_z loners, assuming that each contribution is equal ($c = 1$), are defined by (Hauert et al., 2002a, 2004):

$$p_x = r\frac{n_x}{s} - 1, \quad p_y = r\frac{n_x}{s}, \qquad p_z = \sigma, \tag{3.1}$$

where the parameters r, σ, and n are considered under the following assumptions:

Assumption 1 *The interest rate on the common pool r satisfies $1 < r < n$.*

If $1 < r$, i.e., if all individuals cooperate, they are better off than if all defects. If $r < n$, each individual is better off defecting than cooperating (Hauert et al., 2002a).

Assumption 2 *The payoff of the loner strategy σ satisfies $0 < \sigma < r - 1$.*

This means that the revenue of a cooperator in a group of cooperators (profiting $(r-1)$ each) is better off than loners that receive σ. Still loners are better off than a defector in a group of defectors, where the payoff is equal to 0 (Hauert et al., 2002a). We consider that only a set d $(0 \leq d \leq 1)$ of the defectors will be punished in this work. In particular we assume that this set of randomly selected defectors will have their corresponding payoff reduced to 0, while the remaining free riders will obtain the normal payoff. Therefore, a defector not knowing if she will be punished or not, in the presence of m cooperators will have a payoff equal to:

$$p_y = (1-d)\left(\frac{rn_x}{s}\right) + d0. \tag{3.2}$$

The payoff in (3) and (3) are defined for a particular number of cooperators, defectors and loners. However, in a public goods game, an individual does not know the strategy selected by the group's other members. To compute a player's payoff, consider that this group composition depends on the frequencies of all frequencies in the population. Therefore, in the sample group of size n with s $(s \leq n)$ being the number of those who will participate (cooperators and defectors) and $n-s$ those who refuse (loners). If $s=1$, the game cannot take place, and regardless of the strategy of the only player, the payoff will be equal to a loner payoff; this happens with a probability z^{n-1}.

If $s \geq 2$, an individual that agrees to play has $s-1$ coplayers from the $n-1$ sample given by the probability (Hauert et al., 2002a)

$$\binom{n-1}{s-1}(1-z)^{s-1}z^{n-s}. \tag{3.3}$$

Furthermore, the probability that j of these $s-1$ coplayers will be cooperators and the others $s-1-j$ defectors is (Hauert et al., 2002a)

$$\binom{s-1}{j}\left(\frac{x}{x+y}\right)^{j}\left(\frac{y}{x+y}\right)^{s-1-j}. \tag{3.4}$$

The expected payoff for a defector in a group of $s(s=2,...,n)$ players over all possible numbers of cooperators is:

$$(1-d)\frac{r}{s}\sum_{j=0}^{s-1} j\binom{s-1}{j}\left(\frac{x}{x+y}\right)^{j}\left(\frac{y}{x+y}\right)^{s-1-j} = (1-d)\frac{r}{s}(s-1)\frac{x}{x+y}.$$

Taking into consideration that a defector can be alone $(s=1)$ or in a group $(s \geq 2)$ and that both the number or participants (s) and cooperators (j) in the group are random variables, the average payoff for a defector over all possible numbers of participants is given by:

$$p_y(x,z,d) = \sigma z^{n-1} + (1-d)\left(r\frac{x}{1-z}\sum_{s=1}^{n}\binom{n-1}{s-1}(1-z)^{s-1}z^{n-s}\left(1-\frac{1}{s}\right)\right).$$

Observing that $\binom{n-1}{s-1} = \binom{n}{s}\frac{s}{n}$, then the expression for p_y can be rewritten as

$$\begin{aligned} p_y(x,z,d) &= \sigma z^{n-1} + (1-d)\left(r\frac{x}{1-z}\left(1-\frac{1}{n}\sum_{s=1}^{n}\binom{n}{s}(1-z)^{s-1}z^{n-s}\right)\right) \\ &= \sigma z^{n-1} + (1-d)\left(r\frac{x}{1-z}\left(1-\frac{1}{n}\frac{1-z^n}{(1-z)}\right)\right). \end{aligned}$$

For sake of simplicity, we define an auxiliary function $a := (1-z^n)/(n(1-z))$ then the payoff finally takes the form:

$$p_y(x,z,d) = \sigma z^{n-1} + (1-d)\, r\frac{x}{1-z}(1-a)\,. \tag{3.5}$$

A similar analysis can be performed to obtain the expected payoff of a cooperator in the population. To distinguish the contribution from one cooperator to the contribution of the other cooperators in the group, the payoff can be rewritten:

$$p_x = \left(\frac{r(n_x-1)}{n} + \frac{r}{n}\right) - 1.$$

where $n_x - 1$ (the remaining cooperators in the game) is defined as j.

$$p_x = \left(\frac{rj}{n} + \frac{r}{n}\right) - 1 = \left(\frac{r(j+1)}{n}\right) - 1.$$

Therefore, the expected payoff for a cooperator in a group of s ($s = 2, ..., n$) players over all possible numbers of cooperators is

$$\sum_{j=0}^{s-1}\binom{s-1}{j}\left(\frac{r}{s}(j+1)-1\right)\left(\frac{x}{x+y}\right)^{j}\left(\frac{y}{x+y}\right)^{s-1-j} = \frac{r}{s}(s-1)\frac{x}{1-z} + \frac{r}{s} - 1.$$

If the cooperator is the only one that wants to play, that is, if $s = 1$, the payoff is σz^{n-1}. If ($s \geq 2$), the payoff depends on the number or participants (s) in the group.

Hence, the expected payoff for a cooperator is given by

$$p_x(x,z) = \sigma z^{n-1} + \sum_{s=2}^{n} \binom{n-1}{s-1} (1-z)^{s-1} z^{n-s} \left(\frac{r}{s}(s-1)\frac{x}{1-z} + \frac{r}{s} - 1 \right)$$

$$p_x(x,z) = \sigma z^{n-1} + r\frac{x}{1-z} \left(\sum_{s=2}^{n} \binom{n-1}{s-1} (1-z)^{s-1} z^{n-s} \right) - r\frac{x}{1-z} \left(\sum_{s=2}^{n} \binom{n-1}{s-1} (1-z)^{s-1} z^{n-s} \frac{1}{s} \right)$$
$$+ r \left(\sum_{s=2}^{n} \binom{n-1}{s-1} (1-z)^{s-1} z^{n-s} \frac{1}{s} \right) - \sum_{s=2}^{n} \binom{n-1}{s-1} (1-z)^{s-1} z^{n-s}.$$

Observing that $\binom{n-1}{s-1} = \binom{n}{s}\frac{s}{n}$, we obtain

$$p_x(x,z) = \sigma z^{n-1} + r\frac{x}{1-z}(1-z^{n-1}) - r\frac{x}{1-z} \left(\frac{1}{n}\frac{(1-z^n)}{(1-z)} - z^{n-1} \right)$$
$$+ r \left(\frac{1}{n}\frac{(1-z^n)}{(1-z)} - z^{n-1} \right) - 1 + z^{n-1}.$$

Rearranging and using, as before, the auxiliary function a, we obtain

$$p_x(x,z) = \sigma z^{n-1} + ra + r\frac{x}{1-z}(1-a) + (1-r)z^{n-1} - 1. \tag{3.6}$$

The difference in payoff between defector and cooperator strategies shows the relative benefit (or drawback) of defectors over cooperators, and it is essential to characterize the solution orbits. This difference is given by

$$\begin{aligned} g(x,z,d) := p_y - p_x &= 1 + (r-1)z^{n-1} - \frac{r}{n}\frac{(1-z^n)}{(1-z)} - dr\frac{x}{(1-z)} \left(1 - \frac{1}{n}\frac{1-z^n}{(1-z)} \right) \\ &:= 1 + (r-1)z^{n-1} - ra - dr\frac{x}{(1-z)}(1-a). \end{aligned} \tag{3.7}$$

To highlight the effect of the parameter d, Expression (3.7) is rewritten as

$$g(x,z,d) := m(z) - dxh(z), \tag{3.8}$$

where $m(z) = 1 + (r-1)z^{n-1} - ra$ and $h(z) = \frac{r}{1-z}(1-a)$. Observe that, when $d = 0$, then $g(x,z,d) = m(z)$, and the free system introduced in (Hauert et al., 2002a) is recovered.

For modeling the population dynamics, in this paper and following (Hofbauer and Sigmund, 1998), every individual in the population follows a strategy i (for $i = x, y, z$) from the three options available: cooperators x, defectors y, and loners z. Let $0 \leq x(t), y(t), and z(t) \leq 1$ be the frequency of each of the corresponding available strategies of the population at a specific time t. To simplify the notation, we drop henceforth the time dependency t and simply write x, y, and z. The frequency distribution of the whole population at a specific time t is defined by the state $[x, y, z]$, which belongs to the unit simplex $\mathcal{S}_3$ given by

$$\mathcal{S}_3 = \left\{[x, y, z] \in \mathbb{R}^3 : x, y, z \geq 0 \text{ and } x + y + z = 1\right\}. \tag{3.9}$$

The interior of $\mathcal{S}_3$ is defined as the set of points where all strategies are present (i.e., x $>$ 0, y $>$ 0, and z $>$ 0) and the boundary of $\mathcal{S}_3$ is defined as the set of points that are not interior points. The boundary of the simplex comprises vertices and borders. The vertices x, y, and z represent a homogeneous population of cooperators, defectors, and loners, respectively. In the borders, xy, yz, and zx, one strategy is absent.

Assuming a large enough population in which generations blend continuously into each other, each strategy evolves with a rate that expresses its evolutionary success following the replicator equation

$$\begin{cases} \dot{x} = x\,(p_x - \bar{p}) \\ \dot{y} = y\,(p_y - \bar{p}) \\ \dot{z} = z\,(p_z - \bar{p}) \end{cases} \tag{3.10}$$

with $x(0)$, $y(0)$, and $z(0)$ as initial conditions in the simplex $\mathcal{S}_3$, with $\bar{p} = xp_x + yp_y + zp_z$, where p_i is the payoff of the ith strategy, and $\bar{p}$ is the average payoff of the population.

It is important to remark that the simplex $\mathcal{S}_3$ remains invariant under the flow of Equation (3.10) (see (Hofbauer and Sigmund, 1998; Zeeman, 1980)); hence, if the state $[x, y, z] \in \mathcal{S}_3$ at $t = 0$, then $[x, y, z] \in \mathcal{S}_3$ $\forall t$, and it is endowed with the standard norm $\|\cdot\|_2$. Notice that, in Expression (3.10), an important behavior relies on the payoff p_i of the ith strategy with respect to the average population payoff $\bar{p}$ in the sense that, if $p_i > \bar{p}$, then the proportion of the i strategy increases, otherwise it decreases.

A new variable $f := x/(x+y)$ is defined (Hauert et al., 2002a). This variable represents the fraction of cooperators among the players in the game and allows one to define a bijection $\mathcal{T} : (x, y, z) \to (f, z)$, provided that the restriction $x + y + z = 1$ is satisfied. Hence, the system (3.10) can be rewritten in a more convenient reduced form (Hauert et al., 2002a):

$$\begin{cases} \dot{f} &= -f(1-f)g(f, z, d) \\ \dot{z} &= z(\sigma - \bar{p}(f, z)) \end{cases} \tag{3.11}$$

where $(f,z) \in [0,1]\times[0,1]$. The function $g(f,z,d)$ is obtained replacing $x = f\,(1-z)$ in Expression (3.8), so $g(f,z,d) = 1+(r-1)z^{n-1} - ra - drf\,(1-a)$. To obtain a more explicit expression than (3.11), consider that $y = 1-x-z$, and $f = x/(1-z)$. The system of Equation (3.11) can then be rewritten in the following form:

$$\begin{cases} \dot{f} &= -f(1-f)g(f,z,d) \\ \dot{z} &= z(1-z)\left(\sigma + fg(f,z,d) - p_y(f,z,d)\right). \end{cases} \tag{3.12}$$

For each time t, d parametrizes the solution (f,z) of the system of Equation (3.12). An equilibrium point of System (3.12) exists in the interior of the simplex if the following expressions are satisfied:

$$g(f,z,d) = 0 \tag{3.13a}$$

$$\sigma - fg(f,z,d) - \sigma z^{n-1} - (1-d)rf(1-a) = 0, \tag{3.13b}$$

and since both equations have to be satisfied simultaneously, Equation (3.13b) is reduced to $\sigma(1-z^{n-1}) - (1-d)rf(1-a) = 0$, so the value of f can be obtained by

$$f = \frac{\sigma}{(1-d)r}\frac{(1-z^{n-1})}{(1-a)}. \tag{3.14}$$

By then introducing f in (3.13a), an equivalent definition for $g(\cdot)$ that depends only on z and d is obtained:

$$\tilde{g}(z,d) = m(z) + \tilde{h}(z,d), \tag{3.15}$$

where $\tilde{h}(z,d) = -\frac{d}{1-d}\sigma(1-z^{n-1})$ and $m(z) = 1+(r-1)z^{n-1} - ra$ (as introduced in Expression (3.8)). Now, from (3.15), the value of z in the equilibrium, defined as $\hat{z}$, can be numerically obtained as a function of d. Recalling that a depends only on z and denoting $\hat{a}$ as the expression of a with $\hat{z}$, then (3.13a) and (3.13b) become

$$\begin{cases} 1+(r-1)\hat{z}^{n-1} - r\hat{a} - dr\hat{f}\,(1-\hat{a}) &= 0 \\ \sigma\hat{z}^{n-1} + (1-d)\,r\hat{f}\,(1-\hat{a}) &= \sigma \end{cases} \tag{3.16}$$

where $\hat{f}$ corresponds to the value of f at the equilibrium point associated with $\hat{z}$ for a d. Defining an auxiliary variable as $\tilde{b} := r\hat{a}$, the above equations take the form of a system of two equations with two unknowns ($\hat{f}$ and $\hat{b}$), as follows:

$$\begin{cases} 1+(r-1)\hat{z}^{n-1} - \tilde{b} - d\hat{f}\left(r-\tilde{b}\right) &= 0 \\ \sigma\hat{z}^{n-1} + (1-d)\,\hat{f}\left(r-\tilde{b}\right) &= \sigma \end{cases} \tag{3.17}$$

Extracting $\hat{f} = (\sigma(1-\hat{z}^{n-1})/((1-d)(r-\tilde{b}))$ from the second equation and introducing it in the first equation, $\tilde{b}$ is obtained. Replacing $\tilde{b}$ in the second equation, a new form of $\hat{f}$ in the equilibrium is then obtained:

$$\hat{f} = \frac{\sigma}{(r-1)+d(\sigma-(r-1))}. \tag{3.18}$$

Notice that the interior equilibrium point $(\hat{f}, \hat{z})$ corresponds in the system $(\hat{x}, \hat{y}, \hat{z})$ to the equilibrium point in the interior of the simplex $\mathcal{S}_3$ given by

$$(\hat{x}, \hat{y}, \hat{z}) = \left(\frac{\sigma}{\alpha}(1-\hat{z}), (1-\frac{\sigma}{\alpha})(1-\hat{z}), \hat{z}\right)$$

, where $\alpha = (r-1)+d(\sigma-(r-1))$ and the dependence on the parameter d is given explicitly. To relate both systems, a summary of the equilibrium points concernings to the variables (x, y, z) and their corresponding (f, z) are given in Table 3.1.

Table 3.1: Equivalent equilibrium points between systems $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{f}, \hat{z})$. $\alpha = (r-1)+d(\sigma-(r-1))$, $\beta = (n-r)/(r(n-1)d)$.

$(\hat{x}, \hat{y}, \hat{z})$	$(\hat{f}, \hat{z})$
$(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$	$(\hat{f}, \hat{z}) = (1, 0)$
$(\hat{x}, \hat{y}, \hat{z}) = (0, 1, 0)$	$(\hat{f}, \hat{z}) = (0, 0)$
$(\hat{x}, \hat{y}, \hat{z}) = (0, 0, 1)$	$(\hat{f}, \hat{z}) = (0, 1)$
$(\hat{x}, \hat{y}, \hat{z}) = (\frac{\sigma}{\alpha}(1-\hat{z}), (1-\frac{\sigma}{\alpha})(1-\hat{z}), \hat{z})$	$(\hat{f}, \hat{z}) = (\frac{\sigma}{\alpha}, \hat{z})$
$(\hat{x} = f, \hat{y} = 1-f, \hat{z} = 0)$	$(\hat{f}, \hat{z}) = (\beta, 0)$

The first three equilibrium points in Table 3.1 are the equilibrium points corresponding to the vertex of $\mathcal{S}_3$, the fourth in the interior of the simplex $\mathcal{S}_3$, and the fifth in the border xy. In the next subsections, the effect of d on the equilibrium point will be analyzed. For that reason, $(\hat{f}, \hat{z})$ will henceforth be redefined as $(\hat{f}_d, \hat{z}_d)$ (for $d \neq 0$) to distinguish from the equilibrium when $d = 0$, which will henceforth be defined as $(\hat{f}_0, \hat{z}_0)$.

Before further analysis, let us define another threshold for the parameter d that will be used in the following sections. Notice that Expression (3.15) when $z = 0$ is reduced to $(1 - r/n) - \sigma\,(d/1-d)$, from which $d_2 = (n-r)/(n\sigma + n - r)$ can be obtained.

Chapter 4

Preliminary work

Prior to this work, a literature review of the evolution of cooperation models was performed. The models varied in terms of the number of strategies, type of incentive, sanctioning systems, conditions and size of the population. Six models were chosen, they were compared and systematized in a table (see Table 4.1) following their main characteristics to select those more appropriate to a specific practical case.

Table 4.1: Comparative table. Public goods game models

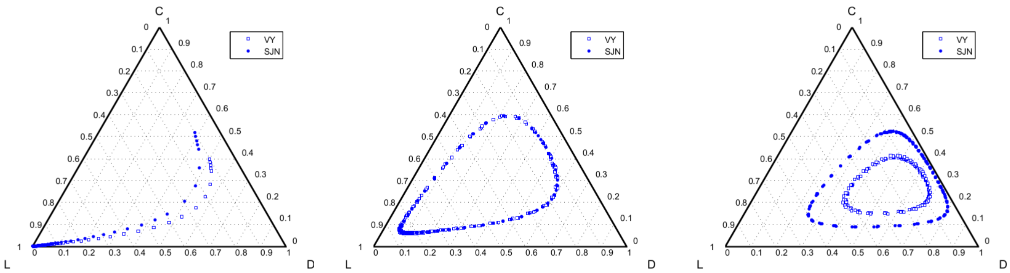
Model	Year	Reference	Game[a]	Strategies [b]	Incentive [c]	Second-order punishment	Population	Conditions
1	2002	Hauert et al.	OPGG	C, D, L	-	-	Large	$r > 1$, $0 < \sigma < r - 1$
2	2005	Fowler	OPGG	C, D, L, P	PeerP	Yes	Large	$r - 1 > \sigma$, $p > c$
3	2006	Brandt et al.	OPGG	C, D, L, P	PeerP	Yes	Large	$n > r > 1 + \sigma$, $\beta > 1 > \alpha > 0$
4	2008	Hauert et al.	OPGG	C, D, L, P	PeerP	Yes	Infinite Finite	$r < n$ $0 \leq \alpha < 1$, $(r - 1)c > \sigma > 0$, $\beta > \gamma$
5	2011	Sasaki and Unemi	CPGG	C, D, R	PoolR	Yes	Infinite	$n \geq 2$, $r_1 < n$
6	2012	Sasaki et al.	OPGG	C, D, L	PoolR PoolP	-	Large	$n \geq 2$,$m \geq 2$, $1 < r < n$

Note. Table from "Evolution of cooperation in evolutionary games for Sanitation Boards" (Botta et al., 2014).
[a] OPGG: Optional Public Goods Game, CPGG: Compulsory Public Goods Game
[b] C: Cooperator, D:Desertor, L:Loner, P:Punisher, R:Rewarder
[c] PeerP: Peer-punishment, PoolP: Pool-punishment, PoolR: Pool-reward

As mentioned before, part of this work was motivated by communities projects. The cooperation, framed within this context, can be seen as the willingness of individuals to support a project that will be beneficial for the community, accepting to pay the cost and obeying the rules. Nevertheless, the temptation to free-ride is always present.

This is the situation in Sanitation Boards. SBs are autonomous community-based organizations, with some interesting particularities, from the way they are created to their administration. The centralized government assists SBs during the early stages of its constitution. Once formed, the SB is responsible for its management. Following (Ostrom, 2002), SBs can be defined as self-governed common-pool resources (CPR). Participants are much involved in the process of establishing, managing and regulating an SB. The project produces finite quantities of resources units but usually the system is greater enough to provide water for the whole community without conflict. Furthermore, excluding beneficiaries is possible but expensive.

The importance of SBs lies in the fact that, altogether, they provide water to 42,6% of the households in Paraguay (MOPC, 2018). SBs have undoubtedly contributed to achieving the Millennium Goals (World Health Organization, 2015) and are a key to the fulfillment of the Sustainable Development Goals (SDG) by 2030. However, some SBs have a high rate of non-payment, a reason that prevent SBs from being successful (Abbate, 2011; MOPC, 2018; Martínez et al., 2004). This situation can be seen as a cooperation problem and, therefore, it could be modeled with a public goods game.



(a) (b) (c)

Figure 4.1: Preliminary work. Sanitation Boards. Model 1. Optional public goods game without incentives. A voluntary game without incentive can sustain cooperation in time when $r > 2$. Parameters: $n = 5$, $\sigma = 1$, **(a)** $r = 2$, **(b)** $r = 3$ and, **(c)** $r = 3,5$. Initial values: San Juan Nepomuceno (SJN) $x_c = 0,52$; $x_d = 0,38$; $x_r = 0,1$ and Villa Ygatimi (VY) $x_c = 0,4$; $x_d = 0,5$; $x_r = 0,1$. *Note.* Figure from "Evolution of cooperation in evolutionary games for Sanitation Boards" (Botta et al., 2014).

From the models in Table 4.1, three (Model 1, Model 5 and, Model 6) were implemented and tested with general data from two SBs obtained in (ABC, 2012; Martínez et al., 2012). Some results are shown in Figures 4.1 - 4.2. The analysis was presented in (Botta et al., 2014) with some suggestions for decision-makers in SBs. After implementing the models, more data from SBs was gathered through a research project (Botta et al., 2017a) to compare field data with the theoretical results. Collecting data was difficult since it may be considered sensitive information for some groups. It made difficult to verify the results from the mathematical model. Although the results obtained were not satisfactory in terms of the number of data collected, it proves very useful to understand the modus operandi and managing problems faced in

SBs.

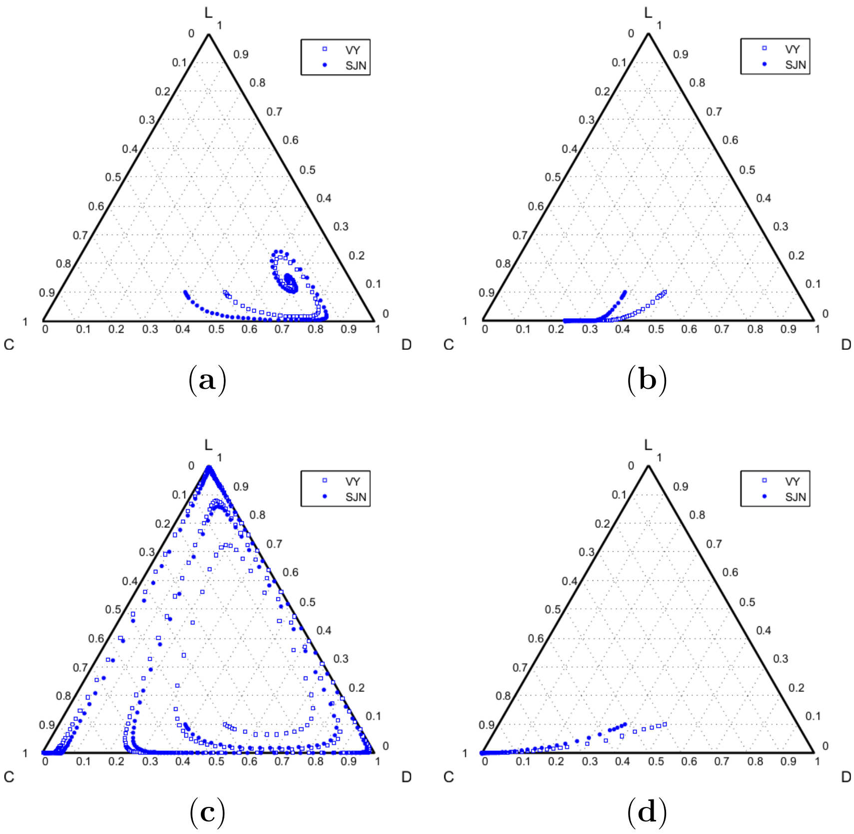


Figure 4.2: Preliminary work. Sanitation Boards. Model 6 ("Self-returning" variant). VPGG with incentives. Positive incentives (**a**) and (**b**) reinforce cooperative behavior and produces coexistence of strategies. Negatives incentives (**c**) and (**d**) produces full cooperation and the process depends on the value of the punishment. In both cases the incentive must be at least moderate for fast and noticeable results. Parameters: $n = 5$, $r = 3$, $c = 1$, $g = 0,5$, (**a**) and (**c**) $I = 0,1$, (**b**) and (**d**) $I = 0,3$. Initial conditions: (SJN) $x_c = 0,52$; $x_d = 0,38$; $x_r = 0,1$ and (VY) $x_c = 0,4$; $x_d = 0,5$; $x_r = 0,1$. *Note.* Figure from "Evolution of cooperation in evolutionary games for Sanitation Boards" (Botta et al., 2014).

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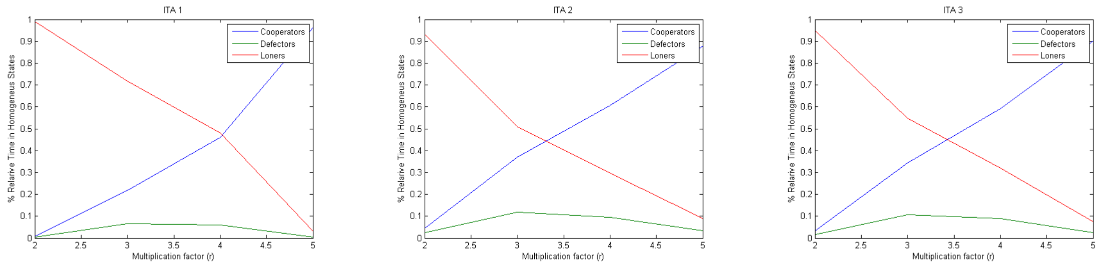
Likewise, exploratory works related to finite population models and control were performed (Botta et al., 2016, 2018, 2017b). In (Botta et al., 2016), the model from (Hauert et al., 2007) was implemented to analyze how the stationary distribution of the system and the probabilities to switch from one state (cooperator, defector, loner or punisher) to the other changes with the size of the population (M), then, in (Botta et al., 2018), the model was implemented with data collected in the reseach project (Botta et al., 2017a) for three SBs (ITA1, ITA2 and ITA3) (see Figure 4.3).

Following what has been observed in these works, it was possible to determine that the studied models are very much adapted for experiments carried out in a laboratory environment and are difficult to apply in real-world situations because they differ in characteristics and procedures. From Table 4.1, an optional public goods game model without incentives is chosen (see Table 4.1 Model 1), and an incentive mechanism oriented to ordinary procedures is added. The next chapter presents the fractional punishment applied to a compulsory public goods game.



(a) (b) (c)

Figure 4.3: Preliminary work. Sanitation Boards. Model (Hauert et al., 2007): Relative time in homogeneous states as a function of the multiplication factor r. Increasing r increment the time the system spend in cooperator state, but the time also depends on the size of the population (M). For bigger SBs r must be larger: To achieve at least 50% of time in cooperator state, ITA1 ($M = 1123$) needs $r \geq 4$, but for ITA2 ($M = 141$) and ITA3 ($M = 205$) with less population $r \geq 3$ is sufficient. Parameters: $n = 5$, $\sigma = 1$, $c = 1$, $s = 0.249$, $b = 1$, $g = 0.3$; (**a**)$M = 1123$; (**b**)$M = 141$; (**c**)$M = 205$. *Note*. Figure from "Finite population evolutionary game for water supply sanitation boards" (Botta et al., 2016).

Chapter 5

Two-strategies fractional punishment

The fractional punishment method was first analyzed in a compulsory public goods game (Botta et al., 2020). Without the possibility to abstain from playing, *i.e.* $z = 0$, there are only two strategies available in the population to cooperate or to defect. The payoff presented in Equations (3.5, 3.6 and, 3.7) are reduced to

$$\begin{aligned} p_y =& (1-d)r\left(1-\frac{1}{n}\right)x \\ p_x =& \frac{r}{n}((n-1)x+1)-1 \\ p_y - p_x =& \left(1-\frac{r}{n}\right)-d\left(r-\frac{r}{n}\right)x =: g(d,x), \end{aligned} \tag{5.1}$$

and, the system 3.10 es reduced to

$$\begin{cases} \dot{x} = x\,(p_x - \bar{p}) \\ \dot{y} = y\,(p_y - \bar{p}) \end{cases} \tag{5.2}$$

In the border xy, $x + y = 1$, thus, the dynamic can be analyzed with one equation:

$$\dot{x} = -x(1-x)g(x,d) = -x(1-x)(1-(r/n)-drx(1-1/n)), \tag{5.3}$$

which equilibrium points are $\hat{x} = 1$ ($\hat{y} = 0$), $\hat{x} = 0$ ($\hat{y} = 1$) and, $\hat{x} = (n-r)/(r(n-1)d)$. The qualitative characterization of the aforementioned equilibrium points were studied with the following lemma (Botta et al., 2020)

Lemma 1 *Let us consider equation (5.3) modeling the border* xy *where* x *denotes the cooperators,* y *the defectors,* $x \geq 0$*,* $y \geq 0$ *and* $x + y = 1$*. Denoting the fraction of defectors being sanctioned by* d*, with* $0 \leq d \leq 1$ *and defining* $d_1 := (n-r)/(r(n-1))$ *then: 1) For* $0 \leq d \leq 1$ *the equilibrium point* $\hat{x} = 0$ *is locally asymptotically stable,* *2) $\hat{x} = 1$ is locally asymptotically stable for $d_1 < d \leq 1$ and 3) the point $\hat{x} = (n - r)/(r(n-1)d)$, that exists in the interior of the border for $d_1 < d$, is an unstable equilibrium point.*

Proof 1 *Consider the Jacobian of the equation (5.3), defined by $J(x, d) = d\dot{x}/dx$ which has the form:*

$$J(x, d) = -(1 - 2x)(1 - (r/n) - drx(1 - 1/n)) + x(1 - x)dr\,(1 - 1/n)\,. \tag{5.4}$$

When the expression (5.4) is evaluated at equilibrium point $\hat{x} = 0$, it takes the form $J(0, d) = -(1-r/n)$. As a consequence, since in the game $r < n$, then the eigenvalue is negative and the equilibrium point is locally asymptotically stable. This also means that the reciprocal equilibrium point $\hat{y} = 1$ is locally asymptotically stable since $x + y = 1$.

If expression (5.4) is evaluated at equilibrium point $\hat{x} = 1$, the Jacobian depends on the value of the parameter d, i.e., $J(1, d) = (1 - r/n) - dr(1 - 1/n)$. The eigenvalue is negative for $d > d_1$, and consequently $\hat{x} = 1$ is locally asymptotically stable; however, if $d < d_1$, then the eigenvalue is positive and $\hat{x} = 1$ is locally unstable.

In order to analyze the equilibrium point $\hat{x} = (n-r)/(r(n-1)d)$, it should be notice that d and x are bounded between 0 and 1. If $d = 1$ then $\hat{x} = (n - r)/(r(n-1))$, this is the minimum value of x for which the equilibrium point exists. Notice also that the maximum value of x is 1 then $d = (n - r)/(r(n - 1))$, this is d_1, the minimum value of d for which the equilibrium point exists. For simplicity, let denote $\hat{x} = (n - r)/(r(n - 1)d)$ as $\tilde{x}$; then, evaluating the expression (5.4) at $\tilde{x}$:

$$J(\tilde{x}, d) = x(x - 1)\left(-dr(1 - 1/n)\right). \tag{5.5}$$

Given that $0 < (1 - 1/n)$ and considering $(n-r)/r(n-1) \leq x < 1$ then the eigenvalue of (5.5) is positive for positive values of d and therefore for $d_1 < d$. Consequently the equilibrium point is unstable.

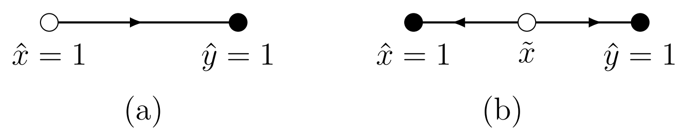


Figure 5.1: Stability of equilibrium points for different values of d: Black dot represents a stable point, white dot represents an unstable point. (**a**) With $d < d_1$, the $\hat{x} = 1$ is unstable and $\hat{y} = 1$ is asymptotically stable. (**b**) If $d_1 < d \leq 1$, an unstable equilibrium $\hat{x} = (n - r)/r(n - 1)d$, denoted by $\tilde{x}$ appears, changing the stability of $\hat{x} = 1$ that becomes an stable equilibrium point

The possible outcomes of the system presented in Lemma 3.1 are sketched in Figure 5.1. It is important to remark the strong dependence of the characterization of the equilibrium points on the function $g(x, d)$. If $0 \leq d < d_1$ then $g(x, d) \neq 0$ for all values of x, and the two equilibrium points are $\hat{x} = 0$ and $\hat{x} = 1$. In such a case, the $\hat{x} = 1$ is globally unstable and $\hat{x} = 0$ is globally asymptotically stable (see Figure 5.1.(a)). For $d_1 < d$ an unstable equilibrium point $\tilde{x}$ exists in the interior of the border (see Figure 5.1 (b)). If $d = (n - r)/r(n - 1)x$ then $g(x, d) = 0$, the corresponding state x is $\tilde{x}$ and it is an equilibrium point.

In Figure 5.2, the outcomes are shown in terms of the payoff of each strategy and the average population payoff since the rule in replicator dynamics is to compare both. When $0 \leq d < d_1$, for all values of x, $p_x < \bar{p} < p_y$, therefore the outcome will be full defection (see Figure 5.2.(a)). For $d_1 < d \leq 1$, the outcome depends on the composition of the population concerning the equilibrium point $\tilde{x}$. As can be seen in Figure 5.2.(b), if $x < \tilde{x}$, then $p_x < \bar{p}$ and the outcome is full defection. However, if $\tilde{x} < x$, then $p_y < \bar{p}$ and the outcome is full cooperation. Since the maximum value of d is 1 there is a minimum value of $\tilde{x}$, see Figure 5.2.(c).

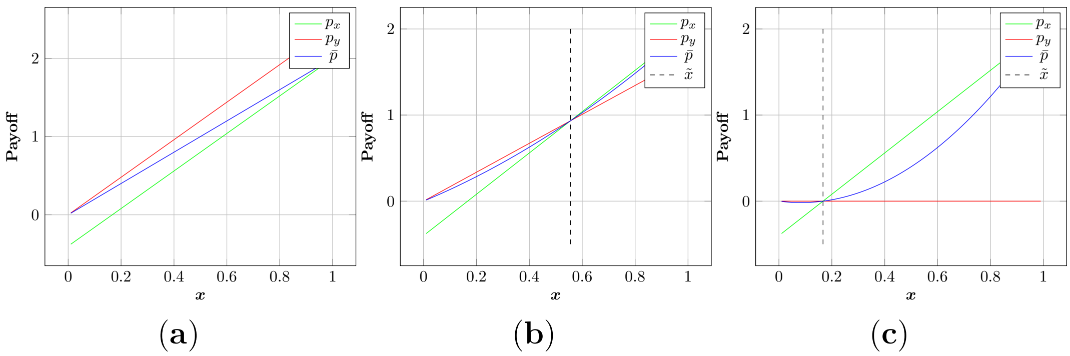


Figure 5.2: Strategies payoff and average population payoff for all possible frequencies of x and different values of d. With replicator dynamics, a strategy increases if its payoff is greater than the average population payoff: (**a**) With $d < d_1$, for all values of x, $p_y > p_x$. Since the payoff of the defectors is greater than the average population payoff ($\hat{P}$) for all values of x, the outcome will be full defection. (**b**) and (**c**) For $d_1 < d \leq 1$, the outcome depends on x; $p_y > p_x$ for $x < \tilde{x}$. ($\tilde{x} = (n - r)/r(n - 1)d$ is indicated by the black dashed line), and $p_y < p_x$ for $x > \tilde{x}$; hence, cooperators decrease if their frequency is smaller than $\tilde{x}$ and increase if it is greater than $\tilde{x}$. Parameters: $r = 3$, $n = 5$, (**a**) $d = 0$, (**b**) $d = 0.3$, (**c**) $d = 1$.

The function $g(x, d)$ represents the difference between defectors and cooperators payoff; its value indicates which strategy is preferred. The function $g(x, d)$ depends on the parameters r, n and, d, but it also depends on the frequency of cooperators in the group (see Figure 5.3).

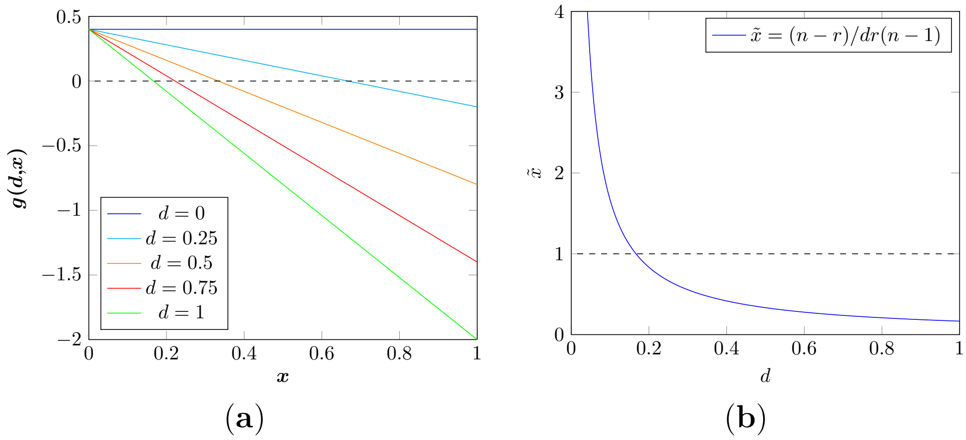


Figure 5.3: **(a)** $g(x, d)$ as a function of x for several values of d: $g(x, d)$ is positive when $d = 0$ for all values of x, for $d > 0$, $g(x, d)$ changes sign when $\hat{x} = (n - r)/(r(n - 1)d)$, **(b)** Relation bewtween $\hat{x}$ and d: as d increases, $\hat{x}$ decreaeses. Since $0 \leq \hat{x} \leq 1$, $\hat{x} \geq 1$ is not possible, conversely, $d \leq (n - r)/(r(n - 1)\hat{x})$ does not affect the system. Parameters: $r = 3$, $n = 5$.

According to the results in a compulsory game, sanctioning only a fraction of the free-riders improves the level of cooperation. This outcome depends on the initial frequency of cooperators in the group (x), particularly, if x is less than a threshold, sanctioning even all defectors does not produce the desired effect and the final state of the system is a population composed only of defectors. If x surpasses the threshold, full cooperation can be achieved. In a project or institution, everyone contributes and the enterprise will be able to prosper. The value of d required to achieve full cooperation depends on the parameters r and n. In addition, the higher the initial frequency of x, the lower the value of d needed to achieve full cooperation.

Sanctioning only a fraction of the free-riders is a pragmatic solution when resources are insufficient. In the next chapter, the model is extended to three strategies, modeling an optional public goods game.

Chapter 6

Three-strategies fractional punishment

To analyze the fractional punishment in optional public goods games, the payoffs presented in Chapter 3 are used. First, in Section 6.1 the boundary of the simplex is analyzed, particularly the borders xz and zy that have not yet been studied. The effect of the parameter d on the values of f and z and the qualitative characterization of the interior equilibrium point. is presented in Section 6.2. In Section 6.3, the equilibrium point representing full cooperation $x = 1$, and its basin of attraction is analyzed. Finally, in Section 6.4, the effect of parameter d on the entire system is resumed.

6.1 Border of the simplex

In the borders, System (3.10) reduces to two equations. The previous chapter studied the border xy; next, the borders zx and yz are analyzed. Observe that the parameter d does not affect the borders zx and yz. Furthermore, since the game setup defines $0 < \sigma < r - 1$, the dynamic of the border yz goes from full defection to only loners in the population, while the dynamic of the border xz goes from only loners in the population to full cooperation.

6.1.1 Border zx

This border represents a population of cooperators and loners. When $x + z = 1$, it is sufficient to analyze the equation $\dot{x} = ((r-1) - \sigma)x(1-x)\left(1 - z^{n-1}\right)$ to characterize the system. If $(r-1) \neq \sigma$, the equilibrium points are $\hat{x} = 1$ $(\hat{z} = 0)$ and $\hat{x} = 0$ $(\hat{z} = 1)$. When $(r-1) = \sigma$, the system remains constant, and every point in the border is an equilibrium point. The Jacobian of $\dot{x}$ has the following form:

$$J(x, z) = (1 - 2x)\Big(((r-1) - \sigma)(1 - z^{n-1})\Big) \tag{6.1}$$

Evaluating (6.1) shows that the equilibrium point $\hat{x} = 1$ ($\hat{x} = 0$) is stable (unstable) if $(r-1) > \sigma$. The dynamic of the border goes from only loners in the population to full cooperation. On the contrary, if $(r-1) < \sigma$, the equilibrium point $\hat{x} = 1$ ($\hat{x} = 0$) is unstable (stable) and the system reverses direction; it goes from full cooperation to a population of loners. When $(r-1) = \sigma$ the system remains constant and every point in the border is an equilibrium point, regardless of the value of z. To observe this consider the ratio between the strategies defined by $\rho := x/z$ (Sigmund, 2010; Weibull, 1997) and notice that its variation on time is equal to 0.

$$\frac{d\rho}{dt} = \frac{\dot{x}z - x\dot{z}}{z^2} = \rho\left(p_x - p_z\right) = \rho\Big((r-1) - \sigma)(1 - z^{n-1})\Big). \tag{6.2}$$

6.1.2 Border yz

This border represents a population composed of defectors and loners. Since $y + z = 1$, the system can be completely represented by $\dot{y} = y(1-y)\sigma(z^{n-1} - 1)$. When $\sigma \neq 0$, the equilibrium points are $\hat{y} = 1$ ($\hat{z} = 0$) and $\hat{y} = 0$ ($\hat{z} = 1$). When $\sigma = 0$, the system remains constant, and every point in the border is an equilibrium point. The Jacobian of $\dot{y}$ has the following form:

$$J(y, z) = (1 - 2y)\sigma(z^{n-1} - 1) \tag{6.3}$$

The equilibrium point $\hat{y} = 1$ ($\hat{y} = 0$), is unstable (stable) if $\sigma > 0$. The dynamic in the border goes from full defection to a population of loners. Conversely, if $\sigma < 0$, $\hat{y} = 1$ ($\hat{y} = 0$) is stable (unstable) and the system reverses direction, going from only loners to full defection. When $\sigma = 0$ the ratio between the strategies cooperator and loner defined by $\rho := y/z$ (Sigmund, 2010; Weibull, 1997) remains equal to 0 in time. Every point in the border is an equilibrium point, this happens regardless of the value of z:

$$\frac{d\rho}{dt} = \frac{\dot{y}z - y\dot{z}}{z^2} = \rho\left(p_y - p_z\right) = \rho\left(\sigma(z^{n-1} - 1)\right). \tag{6.4}$$

6.2 Interior of the simplex

An interior point in the simplex, represents a population where the three strategies are present: cooperators, defectors, and loners. The parameter d modifies the position, nature and number of equilibrium points of the system. Two values of d are especially important in the analysis; the first, $d_1 := (nr)/(r(n-1))$ was defined in the Lemma 5, corresponds to the value of d when $x = 1$ and the second, $d_2 = (nr)/(n\sigma + nr)$, comes from the reduction of the Expression (3.15) when $z = 0$. Both values are used in the following sections.

6.2.1 Effect of d on the value of z in the equilibrium

Here, the effect of parameter d on functions $g(f, z, d)$, $\tilde{g}(z, d)$, and in $\hat{z}$ itself is analyzed. To this end, consider that, for $d_0 = 0$, there is a value of $\hat{z}_0 \in (0, 1)$ such that $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0) = 0$ (see Expressions (3.15) and (3.8)). We claim that the punishment parameter d, when $d > d_0 = 0$, displaces the value $\hat{z}_0$ to another $\hat{z}_d$ such that $\hat{z}_d < \hat{z}_0$ with $\tilde{g}(\hat{z}_d, d) = 0$. To corroborate this affirmation about the effect of d, we introduce the following two lemmas. In all cases, Assumptions 1 and 2 are fulfilled. The prime symbol denotes the function's derivative with respect to its independent variable, for instance, $m'(z) = \frac{dm(z)}{dz}$. We are interested in the relationships between $\tilde{g}(z, d)$ and $\tilde{g}(z, 0) = m(z)$ and between $\tilde{g}'(z, d)$ and $m'(z)$ for a small $d > 0$ and for z in the neighborhood of $\hat{z}_0$ where $z > \hat{z}_0$.

Lemma 2 *Consider $\hat{z}_0 \in (0, 1)$ associated to $d_0 = 0$ such that $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0) = 0$. Let $0 < \epsilon << 1$ such that $\mu = \hat{z}_0 + \epsilon$, and $m(\mu)$, $\tilde{g}(\mu, d)$, $m'(\mu)$, and $\tilde{g}'(\mu, d)$ are negative. Thus, a) $|\tilde{g}(\mu, d)| \geq |m(\mu)|$ and b) $|m'(\mu)| \geq |\tilde{g}'(\mu, d)|$.*

Proof 2 *First part. Consider $\mu = \hat{z}_0 + \epsilon$ with $\epsilon << 1$. Observing expression (3.15) evaluated at z_0, it can be observe that both $m(\mu)$ and $\tilde{h}(\mu) := -d\sigma(1 - \mu^{n-1})/(1 - d)$ are negative for $\mu \in (0, 1)$. Hence $|m(\mu)| < |\tilde{g}(\mu, d)|$ for $d \neq 0$ and $|m(\mu)| = |\tilde{g}(\mu, d)|$ for $d = 0$.*

Second part. Consider $\mu \in (0, 1)$ with $m(\mu) < 0$ and $m'(\mu) < 0$, then

$$\tilde{g}'(\mu, d) = m'(\mu) + (n - 1)\frac{d}{1 - d}\sigma\mu^{n-2} \tag{6.5}$$

Observing that the term $-\frac{d}{1-d}\sigma(1 - \mu^{n-1})'|_\mu = +(n - 1)\frac{d}{1-d}\sigma\mu^{n-2}$ is positive, and since $m'(\mu)$ and $g'(\mu, d)$ are negative, then $|\tilde{g}'(\mu, d)| \leq |m'(\mu)|$.

The effect of the parameter d on the solution $\hat{z}_d$ of $\tilde{g}(z, d) = 0$ concerning the solution $\hat{z}_0$ of $\tilde{g}(\hat{z}_0, d_0) = m(\hat{z}_0) = 0$ of the free system model is analyzed in the following lemma.

Lemma 3 *Considering the values of $\hat{z}_0$ and $\hat{z}_d$ associated to $d_0 = 0$ and $d > d_0$ such that $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0) = 0$ and $\tilde{g}(\hat{z}_d, d) = 0$; hence, $\hat{z}_d < \hat{z}_0$.*

Proof 3 *Consider* $\eta = \hat{z}_0 + \epsilon$ *with* $0 < \epsilon << 1$*. Thus,* $\eta \approx \hat{z}_0$*. Considering the Taylor series expansion truncated at the first order for* $m(\eta)$ *as follows, i.e. high-order terms of the series are neglected, we have*

$$m(\eta - \epsilon) = m(\eta) - m'(\eta)\epsilon. \tag{6.6}$$

Since $\hat{z}_0 = \eta - \epsilon$ *and* $m(\hat{z}_0) = 0$ *(because* $\hat{z}_0$ *satisfies* $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0)$ *and* $\tilde{g}(\hat{z}_0, 0) = 0$*),* $\epsilon = m(\eta)/m'(\eta)$*.*

Consider $\eta = \hat{z}_d + \zeta$*, performing a similar analysis for* $m(\eta-\epsilon)$ *but now for* $\tilde{g}(\eta - \zeta, d)$*. Thus,*

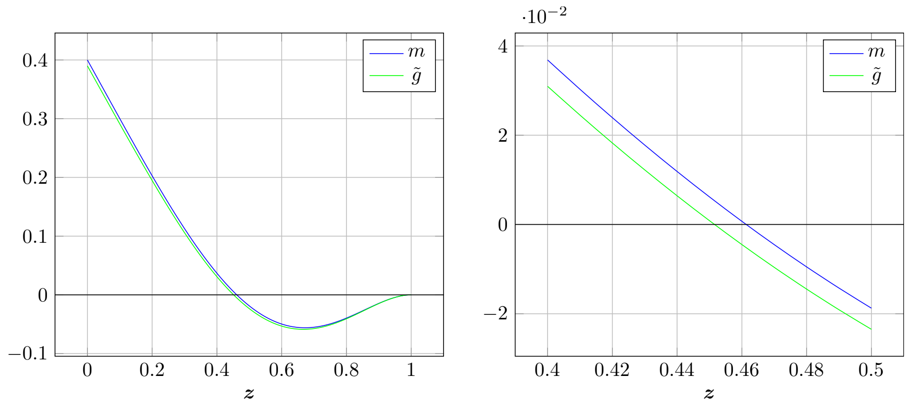
$$\tilde{g}(\hat{z}_d, d) = \tilde{g}(\eta, d) - \tilde{g}'(\eta, d)\zeta = 0 \tag{6.7}$$

and $\zeta = \tilde{g}(\eta, d)/\tilde{g}'(\eta, d)$*. Using the results of Lemma 2, we obtain*

$$|\epsilon| = \frac{|m(\eta)|}{|m'(\eta)|} \leq \frac{|\tilde{g}(\eta, d)|}{|m'(\eta)|} \leq \frac{|\tilde{g}(\eta, d)|}{|\tilde{g}'(\eta, d)|} = |\zeta| \tag{6.8}$$

Since $\hat{z}_0 = \eta - \epsilon$ *and* $\hat{z}_d = \eta - \zeta$*, it is possible to conclude that* $\hat{z}_d \leq \hat{z}$*.*

It is important to mention that, due to the value of d considered being small and the smoothness of $m(z)$ and $\tilde{g}(z, d)$, the high order terms in the expansion in (6.6) do not significantly affect the analysis and can be neglected. The results of Lemmas 2 and 3 can be observed in Figure 6.1, where the functions $m(z)$ and $\tilde{g}(z, d)$ for a $d = 0.01$ concerning z in the interval $(0, 1)$ are presented. The effect of displacing $\hat{z}$ to the left is observed; *i.e.*, $\hat{z}_d$ with $d > 0$ is smaller than $\hat{z}$ with $d = 0$.



(a) (b)

Figure 6.1: (**a**) $m(z)$ and $\tilde{g}(z, d)$ with $d = 0.01$. In the whole interval $z \in (0, 1)$. (**b**) Zoom in the neighborhood of $m(z) = 0$ and $\tilde{g}(z, d) = 0$. It can be observed that d decreases the value of $\hat{z}$ in the equilibria. It shows how d shifts the equilibrium point $\hat{z}$ to the border of $z = 0$. Parameters: $n = 5$, $r = 3$, and $\sigma = 1$.

Figure 6.2 shows the evolution of $\hat{z}_d$ concerning several values of the parameter d (increasing values) remaining constant n, r, and σ. It can be observed that the main effect of the parameter d is precisely to displace $\hat{z}$ to the left from its initial position. This property is exploited for the remainder of the article.

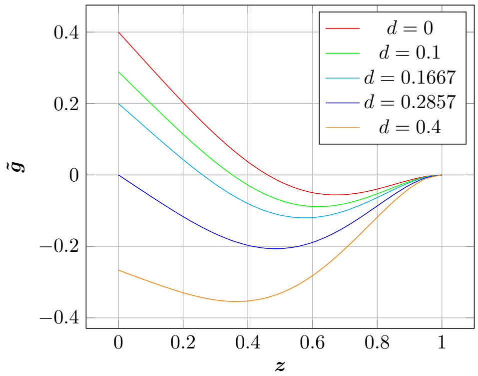


Figure 6.2: Graph of $\tilde{g}$ with several values of d. Observe that, if $z = 0$, $\tilde{g} = 0$ when $d = d_2(0.2857)$. Parameters: $n = 5$, $r = 3$, and $\sigma = 1$.

6.2.2 Effect of d on the value of f in the equilibrium

Here it is analyzed the effect of the parameter d on the value of f in the equilibrium. Recall that at the interior equilibrium point $\hat{f}_d = \frac{\sigma}{(r-1)+d(\sigma-(r-1))}$. Furthermore, due to game setup $\sigma < (r-1)$ (see (Hauert et al., 2002a)). As a consequence, the expression

$$(r-1) + d(\sigma - (r-1)) < (r-1) \tag{6.9}$$

for all $0 < d < 1$. This means that for a $0 \le d < \tilde{d}$ and its corresponding $\hat{f}_0$, $\hat{f}_d$ and $\hat{f}_{\tilde{d}}$, respectively; the following relationship holds

$$\hat{f}_0 = \frac{\sigma}{(r-1)} \le \hat{f}_d = \frac{\sigma}{(r-1)+d(\sigma-(r-1))} < \hat{f}_{\tilde{d}} = \frac{\sigma}{(r-1)+\tilde{d}(\sigma-(r-1))}. \tag{6.10}$$

This expression means that the effect of the parameter d on the equilibrium value $\hat{f}$ consist of incrementing its value or equivalently, improving the level of cooperation. Figure 6.3 shows the changes in the value of $\hat{f}$ and $\hat{z}$ when d increases its value from 0 to d_2. It can be seen how $\hat{f}$ increases while $\hat{z}$ decreases.

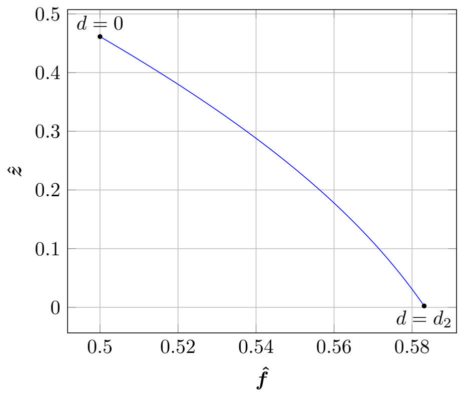


Figure 6.3: Interior equilibrium point ($\hat{f}$ and $\hat{z}$) for increasing values of d. Observe that when d increases; the value of $\hat{f}$ increases while the value of $\hat{z}$ decreases. Parameters: $n = 5$, $r = 3$, $\sigma = 1$ and $d_2 = (n - r)/(n\sigma + n - r)$.

6.2.3 Effect of d on the interior equilibrium point

Using the definition for a (introduced in (B.3)), Equation (3.11) (or equivalently Equation (3.12)) can be rewritten as follows:

$$\begin{cases} \dot{f} &= -f(1-f)\, g(f,z,d) \\ \dot{z} &= z(1-z)\Big(\Big(1-z^{n-1}\Big)\Big(\sigma - f(r-1)\Big) - dfr\,(1-a)\,(f-1)\Big). \end{cases} \tag{6.11}$$

For simplicity, we will drop the arguments in $g(f, z, d)$ and write simply g. To characterize the equilibrium point at the interior of the simplex, we consider the Jacobian of System (6.11), which has the following form:

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

where

$$J_{11} = d\dot{f}/df = -((1-2f)g - f(1-f)dr(1-n)), \tag{6.12}$$

$$J_{12} = d\dot{f}/dz = -f(1-f)((n-1)(r-1)z^{n-2} - rn' + drfn'), \tag{6.13}$$

$$J_{21} = d\dot{z}/df = z(1-z)((1-r)(1-z^{n-1}) - (1-a)dr(2f-1)) \tag{6.14}$$

$$J_{22} = d\dot{z}/dz = (1-2z)((1-z^{n-1})b - e(1-a)) + z(1-z)(-(n-1)z^{n-2}b - e(-a')) \tag{6.15}$$

with $a' = (1 - z^n - nz^{n-1}(1-z))\,/\,(n(1-z)^2)$, $b = \sigma - f(r-1)$ and $e = drf(f-1)$.

In the interior equilibrium equilibrium point, $g = 0$ and $(1 - z^{n-1})b - e(1 -$

$a) = 0$. Evaluating the Jacobian then yields

$$J = \begin{pmatrix} f(1-f)dr(1-a) & -f(1-f)\Big((n-1)(r-1)z^{n-2} \\ & -ra' + drfa'\Big) \\ & \\ z(1-z)\Big((1-r)(1-z^{n-1}) & z(1-z)\Big(-(n-1)z^{n-2}b \\ -(1-a)(2drf-dr)\Big) & -e(-a')\Big) \end{pmatrix}. \tag{6.16}$$

To simplify the analysis, considering the influence of the parameter d, we separate the Jacobian in two matrices J_R and J_T of dimension 2×2 as $J = J_R + d\,J_T$ with

$$J_R = \begin{pmatrix} 0 & -f(1-f)\Big((n-1)(r-1)z^{n-2} - ra'\Big) \\ & \\ z(1-z)\Big((1-r)(1-z^{n-1}) & -z(1-z)(n-1)z^{n-2}(\sigma - f(r-1)) \end{pmatrix} \tag{6.17}$$

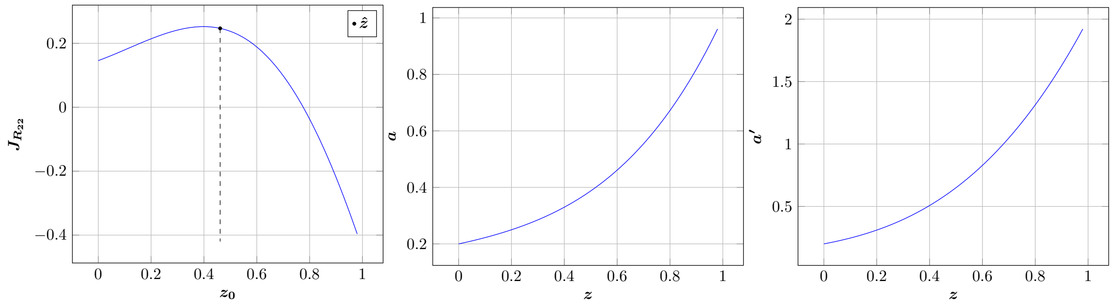
and

$$J_T = \begin{pmatrix} f(1-f)r(1-a)\Big) & -f(1-f)\Big(rfa'\Big) \\ & \\ -z(1-z)\Big((1-a)r(2f-1)\Big) & z(1-z)rf(1-a)(f-1)a' \end{pmatrix}. \tag{6.18}$$

It is important to mention that all the entries of the matrices J_R and J_T have bounded derivatives. If $d = 0$, the Jacobian matrix is reduced to J_R. When (6.17) is evaluated at $(\hat{f}_0, \hat{z}_0)$, it becomes

$$J_R = \begin{pmatrix} 0 & J_{R_{12}} \\ & \\ J_{R_{21}} & 0 \end{pmatrix}. \tag{6.19}$$

Since $\hat{f}_0 = \sigma/(r-1)$ (see Expression (3.18)), $J_{R_{22}} = 0$. The term $J_{R_{21}}$ is negative because $r > 1$. In addition, the term $J_{R_{12}}$ is positive, for $\hat{z}_0 \in (0,1)$ and the parameters n, σ, and r with values normally used in the literature. Figure 6.4.(a) shows the value of $J_{R_{12}}$ for $0 \leq z_0 \leq 1$ and particularly the value in the equilibrium $(\hat{f}_0, \hat{z}_0)$.



(**a**) (**b**) (**c**)

Figure 6.4: Interior equilibrium point analysis, values of $J_{R_{12}}$, a and a' for increasing values of z. (**a**) Values of $J_{R_{12}}$ for $0 \leq z_0 \leq 1$ and $\hat{f}_0 = \sigma/(r-1)$; for $\hat{z}_0$, the term is positive. (**b**) Values of a for $0 \leq z < 1$. (**c**) Values of a' for $0 \leq z < 1$. Parameters: $n = 5$, $r = 3$, and $\sigma = 1$.

The eigenvalues of J_R are given by $\lambda^2 = J_{R_{21}} J_{R_{12}}$, observing that $J_{R_{21}}$ is negative and $J_{R_{12}}$ is positive. The eigenvalues of J_R are then pure imaginary. This result is important for the rest of the analysis of the interior equilibrium point when $d \neq 0$. The fact of that J's eigenvalues are pure imaginary is not conclusive when the interior equilibrium point is a center. However, we are assuming here that $d = 0$, so the system is reduced to the model in (Hauert et al., 2002a), which proves, using a Hamiltonian technique, that the interior equilibrium point is a center.

Next, we analyze the case of the Jacobian matrix J perturbed by the parameter d. To this end, we consider that $d \approx 0$. It is important to remark the implicit dependence of the matrices J_R and J_T on the parameter d.

Theorem 1 *Suppose a Jacobian matrix of System (6.11) denoted by J and decomposed as $J = J_R + dJ_T$ with entries $J_{R_{ij}}$ and $J_{T_{ij}}$ for $i, j \in \{1, 2\}$ with bounded derivatives. Suppose also an initial equilibrium point $(\hat{f}_0, \hat{z}_0)$ of System (6.11) that corresponds to the parameter $d = 0$, and another equilibrium point for $0 < d$ denoted by $(\hat{f}_d, \hat{z}_d)$. Thus, the Jacobian J, evaluated at $(\hat{f}_d, \hat{z}_d)$ with $d \approx 0$, has complex eigenvalues with positive real parts.*

Proof 4 *Consider*

$$J = \begin{pmatrix} 0 & J_{R_{12}} \\ J_{R_{21}} & J_{R_{22}} \end{pmatrix} + d \begin{pmatrix} J_{T_{11}} & J_{T_{12}} \\ J_{T_{21}} & J_{T_{22}} \end{pmatrix} \tag{6.20}$$

Thus, for a positive d and $d = \epsilon \approx 0$, we obtain

$$J = \begin{pmatrix} \varepsilon_{11} & J_{R_{12}} - \varepsilon_{12} \\ \\ J_{R_{21}} \pm \varepsilon_{21} & J_{R_{22}} - \varepsilon_{22} \end{pmatrix} \tag{6.21}$$

where the sign of ε_{ij} is explicitly specified due to the structure of the entries of the matrix J_T. It can be observed that $\varepsilon_{11} > 0$, ε_{12} and ε_{22} are both negative (see Expression (6.18)). The values of a and a' in Figure 6.4.(b-c) and ε_{21} can be positive or negative depending on the value of f on the trajectory of the solution of System (6.11).

The eigenvalues of (6.21) are obtained by

$$(\lambda - \varepsilon_{11})(\lambda - J_{R_{22}} + \varepsilon_{22}) - ((J_{R_{21}} \pm \varepsilon_{21})(J_{R_{12}} - \varepsilon_{12})) = 0. \tag{6.22}$$

Rearranging the expression above, we can obtain the second-order equation in λ

$$\lambda^2 + (-J_{R_{22}} + \varepsilon_{22} - \varepsilon_{11})\lambda - (\varepsilon_{11}(\varepsilon_{22} - J_{R_{22}}) + (J_{R_{21}} \pm \varepsilon_{21})(J_{R_{12}} - \varepsilon_{12})) = 0 \tag{6.23}$$

with a solution

$$\lambda = \frac{1}{2}(J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11}) \pm \frac{1}{2}\sqrt{\Delta} \tag{6.24}$$

where the discriminant $\Delta = (J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2 - 4\,(J_{R_{21}} \pm \varepsilon_{21})(J_{R_{12}} - \varepsilon_{12}) + 4\varepsilon_{11}(\varepsilon_{22} - J_{R_{22}})$.

Analysis of $\psi = J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11}$. First observe that, from Expression (6.10), $\hat{f}_0 < \hat{f}_d$. By Assumptions 1 and 2, and recalling that $J_{R_{22}} = -z(1-z)(n-1)z^{n-2}(\sigma - f(r-1))$, $J_{R_{22}}$ is positive. In addition, comparing ε_{11} and ε_{22} (see that $z(1-z)rf(1-a)(f-1)a' < f(1-f)r(1-a)$ in Expression (6.18)), and considering $d = \varepsilon$, then it is possible to observe that $\varepsilon_{22} < \varepsilon_{11}$, which is controlled by $d = \varepsilon \approx 0$. Hence, the term $J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11}$ is positive.

Analysis of discriminant Δ. Observe that $J_{R_{22}}$ is an analytic function with respect to d (since it depends on $\hat{z}_d$), and $J_{R_{22}} = 0$ for $d = 0$. Considering the Taylor series expansion of $J_{R_{22}} = J_{R_{22}}(0 + d)$ with respect to the parameter d, $J_{R_{22}}$ is given by the expansion

$$J_{R_{22}}(0 + d) = dJ'_{R_{22}}|_{d=0} + \mathcal{O}(d^2)$$

where $J'_{R_{22}}$ denotes the derivative with respect to the parameter d. Observe that d controls $J'_{R_{22}}|_{d=0}$, since choosing arbitrarily d can make the term $dJ'_{R_{22}}|_{d=0}$ as small as *is necessary.*

In addition, the expression $(J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2$ *inside the discriminant takes the form*

$$(J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2 = \Big((dJ'_{R_{22}}|_{d=0} + \mathcal{O}(d^2) - \varepsilon_{22} + \varepsilon_{11}\Big)^2$$

with all terms controlled arbitrarily by d*. As a consequence,* $(J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2 = \mathcal{O}(d^2)$*. In this case, the discriminant* Δ *is*

$$4J_{R_{21}}J_{R_{12}} \pm 4\varepsilon_{21}J_{R_{12}} - 4\varepsilon_{12}J_{R_{21}} \pm 4\varepsilon_{21}\varepsilon_{12} + 4\varepsilon_{11}\varepsilon_{22} - 4\varepsilon_{11}J_{R_{22}} + \mathcal{O}(d^2)$$

where the dominant term is $4J_{R_{21}}J_{R_{12}}$*, since the remaining terms are arbitrarily small. Recalling that* $J_{R_{21}}$ *and* $J_{R_{12}}$ *have opposite signs and the discriminant* Δ *is negative for a sufficiently small* d*. As a consequence, the eigenvalues of (6.21) are complex with a positive real part.*

Theorem 1 shows that, for a small parameter d (considered positive), the eigenvalues of the Jacobian J of System (6.11) are complex with positive real parts when evaluated at the interior equilibrium point $(\hat{f}_d, \hat{z}_d)$. This means that this equilibrium point is an unstable focus; *i.e.*, all solution orbits of System (6.11) are repelled from the equilibrium point to the boundaries. This can be observed in Table 6.1, where some values of d, the corresponding equilibrium points $(\hat{f}_d, \hat{z}_d)$, the entries of the Jacobian J, and its eigenvalues are shown.

For $d = 0$, the eigenvalues of J are pure imaginary, corresponding to the equilibrium point is a center. This result is consistent with the literature (Hauert et al., 2002a). With small positive values of the parameter d ($d = 0.001$ and $d = 0.01$), the eigenvalues have positive real parts, implying that the corresponding equilibrium point is an unstable focus by Theorem 1 (see also Figure 6.5.(a)). It is important to mention that, the equilibrium point remains an unstable focus. Moreover, it is observed that $J_{R_{22}}$ is positive for $0 < d$ and $J_{T_{22}} < J_{T_{11}}$, as mentioned in Theorem 1.

As d increases, the equilibrium point $(\hat{f}_d, \hat{z}_d)$ changes its position from the equilibrium point $(\hat{f}_0, \hat{z}_0)$ with $d = 0$ in the interior of the simplex towards the border xy by decreasing $\hat{z}_d$ and increasing $\hat{f}_d$ (see Sections 6.2.1 and 6.2.2).

For completeness observe that when $d < 0$, the complex eigenvalues of the Jacobian has negative real parts, meaning that the interior equilibrium point is a stable focus (see Figure 6.5.(b). In addition, the equilibrium point is displaced towards the vertex z; the value of $\hat{z}_d$ increases and the value of $\hat{f}_d$ decreases with respect to the equilibrium point values $\hat{f}_0$ and $\hat{z}_0$. Both phenomena can be observed in Table 6.1 for the negative values of the parameter d.

Table 6.1: Effect of the parameter d on the entries of the jacobian J evaluated at the equilibrium point $(\hat{f}_d, \hat{z}_d)$ with the corresponding eigenvalues. For completeness negative values of d are included. Parameters $n = 5$, $r = 3$ and $\sigma = 1$ ($\psi = J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11}$ and $\Delta = (J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2 - 4\,(J_{R_{21}} \pm \varepsilon_{21})(J_{R_{12}} + \varepsilon_{12})$).

d	$J = J_R + d\,J_T$	$(\hat{f}, \hat{z})$	$\lambda = 0.5\,(\psi) \pm 0.5\sqrt{\Delta}$
$d = 0.2857$	$\begin{bmatrix} 0 & 0.14583 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0.58333 & -0.08507 \\ 0 & 0 \end{bmatrix}$	$(0.5833, 0)$	$\lambda_1 = 0$ $\lambda_2 = 0.16667$
$d = 0.2$	$\begin{bmatrix} 0 & 0.21549 \\ -0.32474 & 0.00062 \end{bmatrix} + d \begin{bmatrix} 0.55458 & -0.12910 \\ -0.05102 & -0.02830 \end{bmatrix}$	$(0.5556, 0.2045)$	$\lambda_1 = 0.05199 - 0.24425i$ $\lambda_2 = 0.05199 + 0.24425i$
$d = 0.1$	$\begin{bmatrix} 0 & 0.24964 \\ -0.44938 & 0.00210 \end{bmatrix} + d \begin{bmatrix} 0.51820 & -0.17732 \\ -0.03855 & -0.05327 \end{bmatrix}$	$(0.5263, 0.3524)$	$\lambda_1 = 0.023117 - 0.32244i$ $\lambda_2 = 0.023117 + 0.32244i$
$d = 0.01$	$\begin{bmatrix} 0 & 0.24831 \\ -0.47471 & 0.00046 \end{bmatrix} + d \begin{bmatrix} 0.48162 & -0.21732 \\ -0.00437 & -0.06878 \end{bmatrix}$	$(0.5025, 0.4516)$	$\lambda_1 = 0.00210 - 0.34182i$ $\lambda_2 = 0.00210 + 0.34182i$
$d = 0.001$	$\begin{bmatrix} 0 & 0.24688 \\ -0.47454 & 0.00005 \end{bmatrix} + d \begin{bmatrix} 0.47778 & -0.22112 \\ -0.00044 & -0.06996 \end{bmatrix}$	$(0.5003, 0.4604)$	$\lambda_1 = 0.00021 - 0.34213i$ $\lambda_2 = 0.00021 + 0.34213i$
$d = 0$	$\begin{bmatrix} 0 & 0.24671 \\ -0.47450 & 0 \end{bmatrix} + d \begin{bmatrix} 0.47735 & -0.22155 \\ 0 & -0.07008 \end{bmatrix}$	$(0.5, 0.4613)$	$\lambda_1 = -0.34215i$ $\lambda_2 = +0.34215i$
$d = -0.001$	$\begin{bmatrix} 0 & 0.24654 \\ -0.47445 & -0.00005 \end{bmatrix} + d \begin{bmatrix} 0.47692 & -0.22197 \\ 0.00044 & -0.07021 \end{bmatrix}$	$(0.4998, 0.4623)$	$\lambda_1 = -0.00021 - 0.34216i$ $\lambda_2 = -0.00021 + 0.34216i$
$d = -0.01$	$\begin{bmatrix} 0 & 0.24486 \\ -0.47380 & -0.00052 \end{bmatrix} + d \begin{bmatrix} 0.47305 & -0.22573 \\ 0.00443 & -0.07130 \end{bmatrix}$	$(0.4975, 0.4709)$	$\lambda_1 = -0.00206 - 0.34217i$ $\lambda_2 = -0.00206 + 0.34217i$
$d = -0.1$	$\begin{bmatrix} 0 & 0.21802 \\ -0.45005 & -0.00782 \end{bmatrix} + d \begin{bmatrix} 0.43283 & -0.26134 \\ 0.04539 & -0.07859 \end{bmatrix}$	$(0.4762, 0.5493)$	$\lambda_1 = -0.01876 - 0.33133i$ $\lambda_2 = -0.01876 + 0.33133i$
$d = -0.2$	$\begin{bmatrix} 0 & 0.17097 \\ -0.39826 & -0.02072 \end{bmatrix} + d \begin{bmatrix} 0.38571 & -0.29658 \\ 0.08823 & -0.07940 \end{bmatrix}$	$(0.4545, 0.6238)$	$\lambda_1 = -0.03362 - 0.30224i$ $\lambda_2 = -0.03362 + 0.30224i$
$d = -0.2857$	$\begin{bmatrix} 0 & 0.12005 \\ -0.34223 & -0.03420 \end{bmatrix} + d \begin{bmatrix} 0.34400 & -0.32326 \\ 0.11880 & -0.07492 \end{bmatrix}$	$(0.4375, 0.6799)$	$\lambda_1 = -0.04327 - 0.26828i$ $\lambda_2 = -0.04327 + 0.26828i$

6.3 The equilibrium point $(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$

This equilibrium represents a homogeneous population composed exclusively of cooperators. When $d = 0$, this equilibrium point is a saddle point [1] (Hauert et al., 2002b). Now, we characterize this equilibrium point for $0 < d$ and particularly for $d_1 < d$. For simplicity, the system (f, z) (see Expression (3.11)) will be used. As can be seen in Table 3.1, the equilibrium point $(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$ corresponds to $(\hat{f}, \hat{z}) = (1, 0)$. Evaluating the Jacobian at this equilibrium point is obtained:

$$J = \begin{bmatrix} 1 - r/n - dr\,(1 - 1/n) & 0 \\ 0 & \sigma - (r - 1) \end{bmatrix}. \tag{6.25}$$

[1] Depending on the trajectories, a saddle point equilibrium acts as an atractor or a repulsor. In this work it is considered an unstable equilibrium point

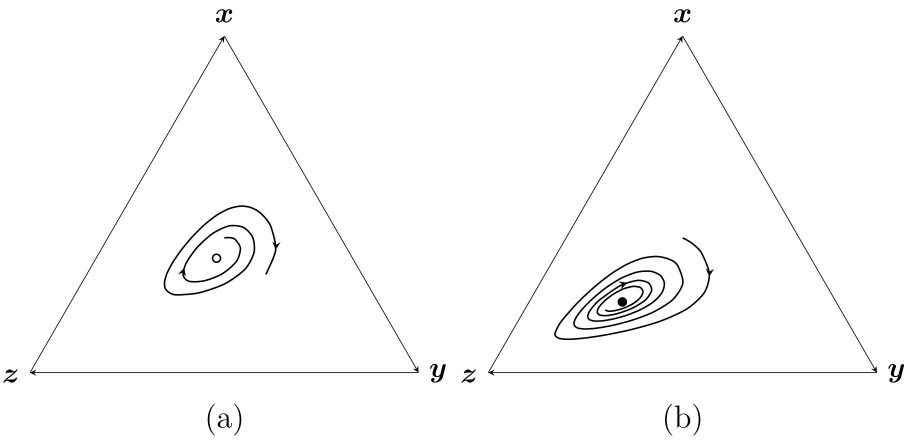


Figure 6.5: Effect of d on the interior equilibrium point for positive and negative d. Parameters $n = 5$, $r = 3$ and $\sigma = 1$, and values of d as (a) $d = 0.1$ and (b) $d = -0.1$. Black dot represents a stable point, white dot represents an unstable point.

The eigenvalues of (6.25) are as follows: $\lambda_1 = 1 - r/n - dr\,(1 - 1/n)$ and $\lambda_2 = (\sigma - (r-1))$. Notice that λ_2 is negative given the condition in the game that $0 < \sigma < (r-1)$ (Assumption 2) and the choice of d determines the sign of λ_1. Recall that $d_1 = (n-r)/r(n-1)$. Thus, for $d < d_1$, λ_1 is positive, the equilibrium point is unstable (a saddle); however, if $d_1 < d$, λ_1 is negative, the equilibrium point is locally asymptotically stable.

To better understand this equilibrium point and its basin of attraction, let us consider the relative-entropy function used in (Weibull, 1997; Sandholm and Ansell, 2010) as a candidate Lyapunov function. To this end, consider the concept of carrier or support of a given state, as the set of pure strategies with positive probability in that state.

Theorem 2 *Suppose that $(\hat{f}, \hat{z}) = (1, 0)$ and the candidate Lyapunov function $V_{(\hat{f},0)} = \hat{f} \ln\left(\hat{f}/f\right)$ such that $V_{(1,0)} > 0$ for all $(f, z) \neq (\hat{f}, 0)$ and $V_{(\hat{f},0)} = 0$ for all $(f, z) = (\hat{f}, 0)$. If*

$$df > \frac{n(1-z)(1+(r-1)z^{n-1}) - r(1-z^n)}{n(1-z)r - r(1-z^n)} \tag{6.26}$$

then $\dot{V}_{(\hat{f},0)} < 0$, and the equilibrium point is locally asymptotically stable

Proof 5 *In the equilibrium point $(\hat{f}, \hat{z}) = (1, 0)$, only f has positive probability, therefore, by using the concept of carrier introduced above, the candidate Lyapunov function takes the form:*

$$V_{\hat{f}}(f) = \hat{f} \ln\left(\frac{\hat{f}}{f}\right), \tag{6.27}$$

with $V_{\hat{f}}(f) > 0$ *for all* $(f, z) \neq (\hat{f}, \hat{z})$ *and* $V_{\hat{f}}(f) = 0$ *for* $(f, z) = (\hat{f}, \hat{z})$.

The derivative with respect of time of this candidate Lyapunov function is given by

$$\dot{V}_{\hat{f}}(f) = -\frac{1}{f}\,\dot{f}. \tag{6.28}$$

Substituting $\dot{f}$ *given by expression (3.12) in the above expression (6.28), it is obtained:*

$$\dot{V}_{\hat{f}}(f) = (1 - f)g(f, z, d), \tag{6.29}$$

where $\dot{V}_{\hat{f}}(f) = 0$ *in the equilibrium point* $(\hat{f}, \hat{z}) = (1, 0)$ *and negative when* $g(f, z, d) < 0$. *To obtain a condition for the parameter* d*, the explicit form of* $g(f, z, d)$ *(see Equation (3.11)) is replaced in equation (6.29), then:*

$$\dot{V}_{\hat{f}}(f) = (1 - f)\left(1 + (r - 1)z^{n-1} - ra - drf(1 - a)\right). \tag{6.30}$$

Observe that $\dot{V}_{\hat{f}}(f)$ *is negative if:* $d > (1 + (r - 1)z^{n-1} - ra) \,/\, (r(1 - a)f)$, *or equivalently,*

$$df > \frac{n(1 - z)(1 + (r - 1)z^{n-1}) - r(1 - z^n)}{n(1 - z)r - r(1 - z^n)} \tag{6.31}$$

In addition $\dot{V}_{\hat{f}}(f)$ *is negative along the trajectories specified by system (3.12), when (6.31) holds, then the equilibrium point* $(\hat{f}, \hat{z}) = (1, 0)$ *is locally asymptotically stable. This means that* $(\hat{f}, \hat{z}) = (1, 0)$ *and the equivalent point in* $\mathcal{S}_3$, $(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$ *is locally asymptotically stable.*

In Figure 6.6, the condition (6.26) is analyzed for increasing values of d, with the restriction $d > d_1$, since this is a requisite for the equilibrium point to be stable (see Lemma 5). There are two main regions in the simplex. The region colored in orange contains all the states where $\dot{V}_{\hat{f}} < 0$, while the region colored in purple contains all the states where $\dot{V}_{\hat{f}} > 0$. Both regions are separated by a red line that divides the simplex from a point in the border xy (where $z = 0$) defined as p_1 to another in the border yz (where $x = 0$) defined as p_2 (see Figure 6.6). This division corresponds to the set of points where $\dot{V}_{\hat{f}}$ changes sign.

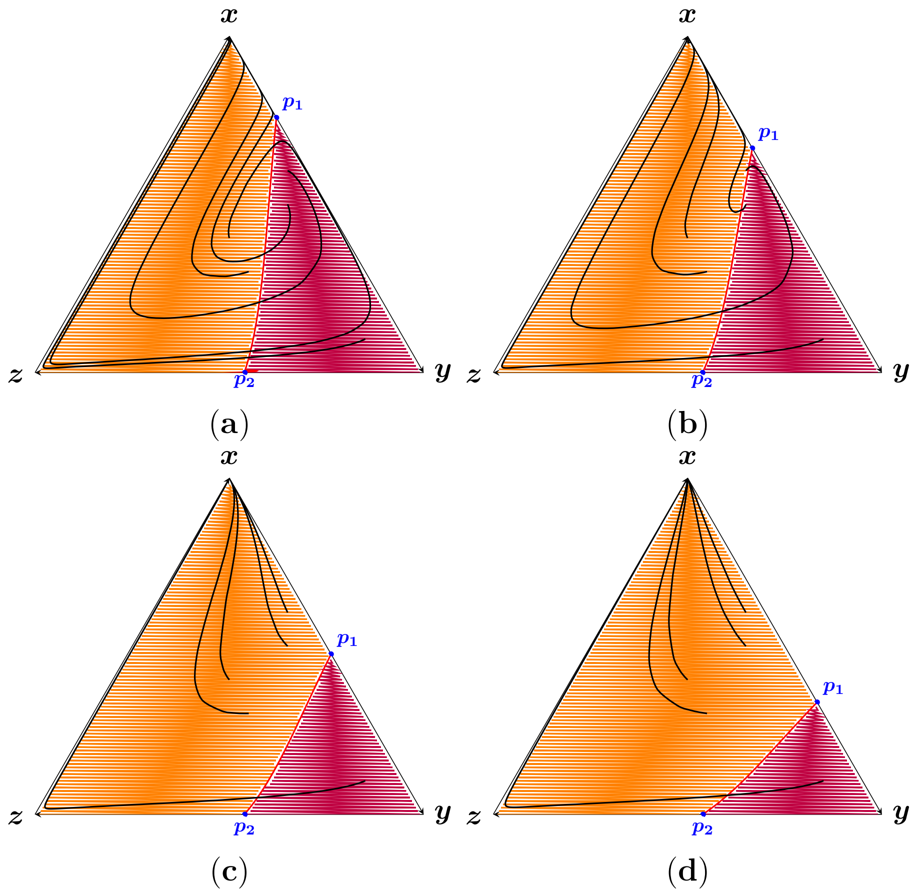


Figure 6.6: Values of $\dot{V}_{\hat{f}}$ for each possible state of the system. The orange area represents states $\dot{V}_{\hat{f}} < 0$; the purple area represents states $\dot{V}_{\hat{f}} > 0$. The red line between both regions represents the set of points where $\dot{V}_{\hat{f}}$ changes sign. The black lines are trajectories from several initial values in the simplex. Parameters $n = 5$, $r = 3$, and $\sigma = 1$, and increasing values of d: **(a)** $d = 0.22$, **(b)** $d = 0.25$, **(c)** $d = 0.35$, and **(d)** $d = 0.5$.

As d increases the region where $\dot{V}_{\hat{f}} < 0$ also increases. This happens because the point p_1 moves to the vertex $y = 1$. Notice that the value of p_1 obtained from $\dot{V}_{\hat{f}} = 0$ is $x = (n-r)/r(n-1)d$. This is the position of the unstable equilibrium in the border xy analyzed in chapter 5. The point p_2, however, does not depend on d. When $x = 0$, the expression is reduced to $1 + (r-1)z^{n-1} - ra = 0$ (see (3.8)). Therefore, the value of z in the point p_2 is fixed and corresponds to the value of $\hat{z}$ when $d = 0$ as in (Hauert et al., 2002a).

Additionally, in Figure 6.6, some paths with initial value points from both regions—$\dot{V}_{\hat{f}} < 0$ and $\dot{V}_{\hat{f}} > 0$ are shown. Due to the structure of the solution, even trajectories with initial values outside the basin of attraction are drawn inside and eventually end in the equilibrium point. In such a case, the equilibrium point $(1, 0, 0)$ is globally asymptotically stable.

Notice in Figure 6.6.(a) that some trajectories oscillate before reaching the equilibrium. This is caused by the unstable equilibrium point in the interior of the simplex analyzed in Section 6.2. Recall that the interior equilibrium shifts to the border xy as d increases, reaching it when $d = d_2$, which, for the parameter in Figure 6.6, is $d_2 = 0.2857$.

From a practical point of view, we are interested in trajectories where the frequency of cooperators does not decrease and, consequently, in the set of initial values from which the number of cooperators increases along all the trajectory to the equilibrium point. To observe this behavior, two regions are defined in the simplex $\mathcal{S}_3$; we denote by Ω a set of initial values where the frequency of the cooperator strategy in the solution orbits continuously grows until full cooperation is reached, i.e.,

$$\Omega := \Big\{(x(t_0), y(t_0), z(t_0)) : x(\tilde{t}) \leq x(t)\Big\} \quad \forall t_0 \leq \tilde{t} \leq t. \tag{6.32}$$

In Figure 6.7, the region Ω is in green. It can also be observed (and this is an important consideration) that Ω is enlarged as the parameter d increases. In addition, the set of initial values in the simplex $\mathcal{S}_3$, where the frequency of cooperators decreases before reaching the equilibrium point, *i.e.* $\mathcal{S}_3 - \Omega$, is colored in blue.

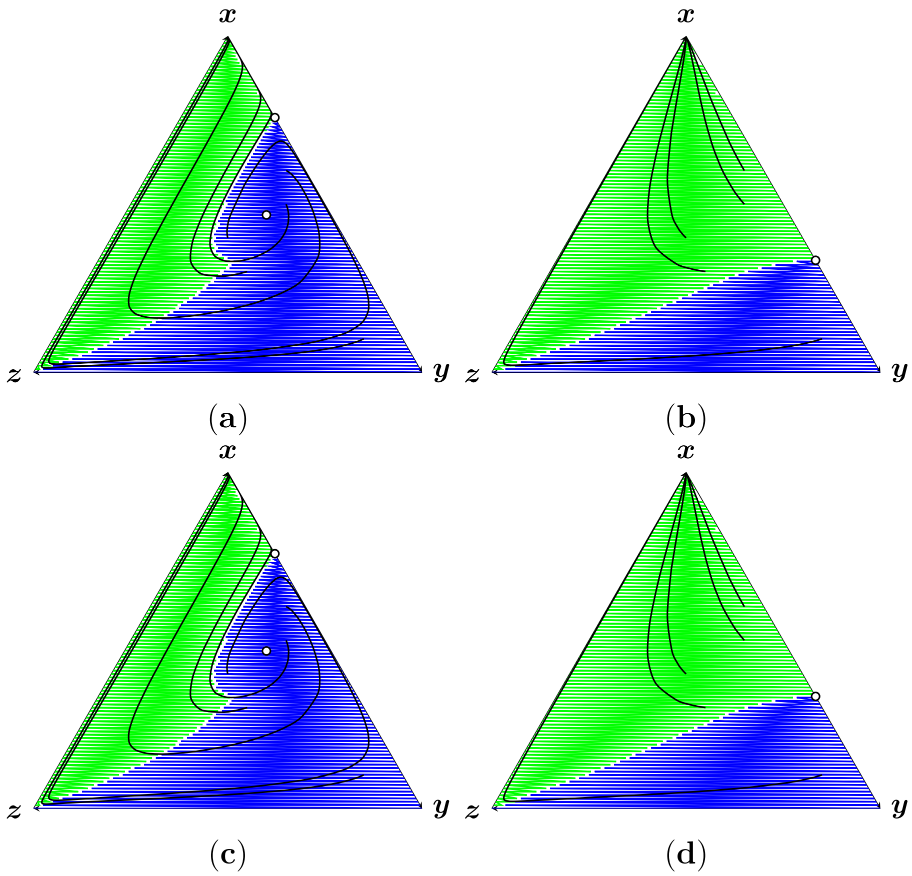


Figure 6.7: The behavior of x in the trajectories for each possible initial value. The green area represents initial values where the frequency of cooperators x increases along the orbit; the blue area represents initial values where at some point of the trajectories x decreases. The black lines are trajectories from several initial values in the simplex. $n = 5$, $r = 3$, and $\sigma = 1$, and values of d are (**a**) $d = 0.22$ and (**b**) $d = 0.5$.

6.4 Effect of the fractional punishment on the system. Summary.

Notice that the entire effect of the parameter d on the dynamic and the equilibrium points of System (3.10) can be described using the thresholds d_1 and d_2 as introduced in Section 6.2.

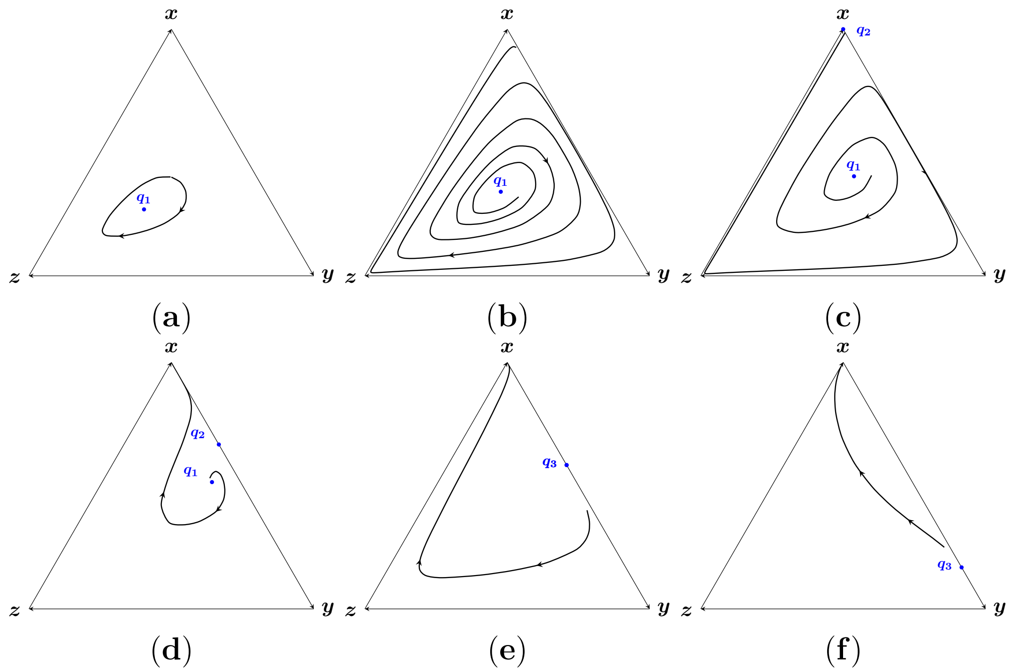


Figure 6.8: Phase diagram of the system for increasing values of d. Parameters $n = 5$, $r = 3$, and $\sigma = 1$. For these values, $d_1 = 0.16667$ and $d_2 = 0.2857$. Notice that q_1, q_2, and q_3 are equilibrium points. **(a)** $d = 0$, **(b)** $d = 0.1$, **(c)** $d = d_1$, **(d)** $d = 0.25$, **(e)** $d = d_2$, and **(f)** $d = 0.99$.

When $d = 0$, the interior equilibrium point q_1 is a center and the the vertices are saddle points and, for this reason, the boundary of the simplex is a rock-scissors-paper type heteroclinic cycle (Hauert et al., 2002a). Figure 6.8.(a) shows the periodic orbits around the equilibrium. Positive values of d cause two transformations in the system. First, the equilibrium point q_1 becomes unstable; second, the position of the equilibrium changes. It moves towards the border xy. Figure 6.8.(b) shows how, with small values of d, $d < d_1$, the trajectories approaching the border of the simplex oscillate.

As d increases, i.e., $d_1 < d < d_2$, a new equilibrium point q_2 arises in the border xy. This is an unstable equilibrium point that moves to the vertex y as d increments its value. Furthermore, when q_2 appears, the vertex x changes its nature from unstable (saddle point) to stable (see Figure 6.8.(c-d)). Notice that the number of oscillations needed to reach the vertex x decreases as d grows. When $d = d_2$, the equilibrium point q_1 reaches the border xy and merges with the equilibrium point q_2 in the border. This new equilibrium q_3 remains unstable (see Figure 6.8.(e)). From $d_2 < d$ to $d = 1$, there is no equilibrium in the interior of the simplex. The point q_3 continues to move toward the vertex y, but it does not reach the vertex; when $d = 1$, the equilibrium point on the xy border takes the value $\hat{x} = (n - r)/r(n - 1)$, this corresponds to the minimum frequency of cooperators required to achieve full cooperation in a compulsory game (see Chapter 5).

Chapter 7

Discussion

Public goods games, both, in the classic compulsory variant where individuals must participate in the game and, with the optional variant where participating is a personal decision, assume conditions that are not easily found in the real-world beyond experiments carried out, in a controlled way, in a laboratory environment. If, in addition, it is decided to study these games from the evolutionary point of view with dynamics such as, for example, the replicator dynamics, the possibility of finding a practical case where all the model requirements are fulfilled, is even lower. However, the problems that these models represent are part of the daily life of all individuals and, even when the theoretical model requirements are not met, many of the results observed can be related to situations that occur in all kinds of institutions and projects.

This duplicity between the theoretical results and its possible practical explanation is interesting and intriguing. Several theoretical results make sense in reality: (1) as mentioned earlier, increasing the cost of service because resources are insufficient, means punishing those who are fulfilling their obligation and easily may incentivize more people to break the rules, (2) the characteristic cycle of strategies found in an optional game variant is common in many projects where a successful stage can lead to a bad situation due to the increase in free-riding to the point that a project can disappear, or (3) as can be seen in Figure 6.8 sanctioning a fraction of the free-riders decreases the percentage of loners in the population at equilibrium, a consequence that seems reasonable since a project where the rules are followed, obtains better results and gives greater confidence in the population to participate in it.

To bring the models closer to real situations, this paper presents a mechanism for applying sanctions widely used in all kinds of institutions, which is to punish only a fraction of the free-riders. To design the model, we consider two common situations regarding the administration of institutions: (1) not all those who did not pay their fees can be punished due to lack of resources; (2) both, the service provided and the cost of the sanctioning system come from the fee paid by the users (fee-pool).

Applying fractional punishment, in the theoretical model, increases the level of cooperation. Even the cooperation of the entire population is achieved if conditions are met. The value of d also defines the trajectory of the solution; with a small d full cooperation is reached after increasing oscillations; when d is greater the oscillations decrease. Of special interest are the trajectories that produce increasing values of cooperation until all the population cooperates. In addition, this sanctioning methodology prevents second-order defectors, that is, cooperators who do not contribute to the sanctioning system, from appearing, avoiding the cost of the second-order sanction. Furthermore, the total cost of sanctioning can be decided in advance, since it depends on the number of defectors to be sanctioned.

Now, if a situation similar to the one modeled is considered in the real-world, and especially a case such as community projects where resources may be scarce, fractional punishment is a useful tool to increase the level of cooperation at a lower cost. Furthermore, since the sanction cost is related to the fraction of sanctioned individuals, it can be adapted to the resources available at that moment in the project. However, with this approach, the pressure to use correctly the total contribution (fee pool) is higher, reaching the right balance between the service expenditure with the expenditure to sanction free-riders (in order to increase cooperation again and, consequently, resources) is more important. Sanctioning free-riders improves cooperation but decreases the profit from the game. We consider that an adequate balance between both expenses is perhaps one of the characteristics of a successful institution.

Chapter 8

Conclusion and future work

The effect of randomly punishing a percentage of defectors in a public goods game modeled using replicator dynamics was analyzed. To this end, the model and the payoff of the strategies were presented, and the equilibrium points in the system were characterized. For the equilibrium point that represents full cooperation, the basin of attraction was studied. Finally, the relation between the simulation results and a possible practical use in community projects was discussed. Future works include fractional sanctioning with redistribution; *i.e.*, the redistribution of the amount collected from the fraction of punished defectors to the cooperators in the group and graded sanctions. The effect of the portion of the fee-pool aimed at sanctioning on the level of cooperation achieved will also be studied. An adaptive sanction that would be more efficient and that accompanies changes in the composition of the group over time will also be considered.

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Anexo

Resumen extendido en español

Capítulo A

Introducción

Esta tesis presenta el modelo de sanción o castigo fraccionado como uno de los posibles mecanismos que incrementan el nivel de cooperación en un grupo de individuos. Este trabajo se realiza en el contexto de la Teoría de Juegos Evolutivos (TJE) que estudia la evolución de comportamientos o rasgos en una población y, particularmente, en base a modelos que analizan la aparición y el mantenimiento de la cooperación, utilizando los juegos de bienes públicos.

La inspiración proviene de prácticas comunes, aunque no específicamente descriptas o modeladas, en instituciones, proyectos comunitarios y grupos de individuos organizados, en donde la morosidad, un comportamiento no cooperativo, se ha convertido en un problema. La morosidad afecta el rendimiento y la supervivencia de un proyecto hasta el punto de que se comienza a sancionar a los que infringen la norma. Sin embargo, sancionar conlleva un costo para el proyecto porque se utilizan recursos (personal, tiempo y dinero) y el grupo puede enfrentarse a la decisión entre sancionar a los morosos o absorber la pérdida. No existe una sola receta de cómo sancionar a los morosos, diferentes grupos han encontrado diferentes soluciones. Una de ellas es sancionar únicamente a un grupo reducido de ellos; este método tiene el beneficio de reducir costos y un ejemplo de esta metodología son los controles al azar. Los mismos tienen como finalidad disuadir al resto de la población de romper las normas impuestas.

La teoría de juegos evolutivos y los modelos de evolución de la cooperación aportan una visión diferente al problema de la morosidad; por ejemplo, en un proyecto con morosidad elevada, en donde los recursos no alcanzan para sostener el sistema, puede decidirse elevar el precio del servicio o el aporte individual para cubrir los costos del funcionamiento. Aumentar el aporte cumple con el objetivo de mantener el servicio pero es contraproducente a largo plazo cuando se analiza desde la evolución de la cooperación como un comportamiento presente en la población. Incrementar el aporte es una forma de sancionar a los que sí cumplen con las normas porque el beneficio del que aporta se vuelve menor. Así, inadvertidamente se estimula el incremento de morosos en la población.

En la literatura, para ejemplificar una situación en la que dos individuos deben decidir si cooperan o no entre sí, se utiliza el dilema del prisionero. Al desconocer, cada uno de ellos, cómo actuará el otro, la opción racional es no cooperar y el resultado es que ambos terminan en una posición peor que si cooperaban entre sí. Cuando más de dos personas participan, a este juego se lo conoce como juego de bienes públicos. En este caso, al igual que en el dilema del prisionero, el resultado final, no cooperar, lleva al grupo a una peor situación que si cooperaran entre sí.

En la evolución de la cooperación, se estudian aquellos mecanismos que favorecen la aparición y mantenimiento de la cooperación en una población. Sin duda, el mecanismo más conocido y aplicado para incrementar la cooperación es utilizar incentivos, sean estos positivos (recompensas) o negativos (sanciones). Otro mecanismo estudiado es la posibilidad del individuo de abstenerse de participar, dando lugar a un juego de bienes públicos opcional. Es esta tesis se analizan los resultados para ambos casos, con énfasis en el juego de bienes públicos opcional que presenta mayor variedad de situaciones y resultados. Este mecanismo más que incrementar la cooperación, facilita su persistencia en el tiempo.

De entre las dinámicas utilizadas para estudiar la evolución de un comportamiento dentro de una población, la más conocida es la dinámica del replicador. De acuerdo a esta dinámica, un comportamiento o estrategia aumenta su frecuencia si el beneficio para los individuos que la utilizan sobrepasa al beneficio promedio de la población; al contrario, si el beneficio obtenido es menor, su frecuencia disminuye y tiende a desaparecer. La intuición detrás de este proceso se puede relacionar con la tendencia de los individuos a comparar constantemente su comportamiento y el beneficio que obtiene con el comportamiento y beneficio que obtienen otras personas imitando aquel comportamiento que provean mayor beneficio.

El modelo de sanción fraccionada presentado en esta tesis es teórico, sin embargo, el comportamiento del sistema es semejante a lo observado empíricamente en casos reales como instituciones y agrupaciones de individuos que trabajan en un proyecto común. Gran parte de este trabajo fue elaborado teniendo como caso de estudio un sistema de provisión de agua potable administrado por la comunidad, las Juntas de Saneamiento (JS) del Paraguay. Muchas JS tienen como principal problema para subsistir su alta tasa de morosidad. Como consecuencia, muchas presentan comportamientos cíclicos de buen funcionamiento inicial, seguido por una disminución de los aportes, o dicho de otra forma, de la cooperación que conlleva a un funcionamiento deficiente y casi desaparición del sistema. Ese comportamiento cíclico es muy similar al resultado encontrado en un juego de bien público opcional. Por otra parte, las JS, con frecuencia se enfrentan a limitaciones tanto de recursos como sociales para sancionar a todos aquellos usuarios que infringen las normas.

La propuesta de esta tesis es modelar la sanción fraccionada o parcial en juegos de bienes públicos. Para alcanzar este resultado, se realizaron trabajos previos incluidos aquí en orden cronológico. Por lo tanto, la organización del trabajo es la siguiente: En la Sección A.1 se presentan los objetivos del trabajo. En el Capítulo A.2 se encuentra la revisión bibliográfica. El área de investigación es altamente interdisciplinaria; por lo tanto, este capítulo contiene una breve reseña de su origen y de sus principales componentes. Asimismo, se presentan los trabajos claves en el área y los trabajos más recientes que sirvieron de base a esta tesis. En el capítulo B, se presentan los conceptos y la metodología común a los capítulos siguientes a fin de facilitar la lectura y no duplicar contenido.

El Capítulo C, contiene los resultados de la primera etapa de la investigación que constituyen los antecedentes de este trabajo. En el Capítulo D, se define el modelo de sanción fraccionada para dos estrategias, esto es, el juego de bienes públicos compulsorio. Este trabajo fue enfocado desde la visión del problema de la morosidad en las Juntas de Saneamiento y fue presentado en el conferencia CCIS (Botta et al., 2020). El modelo de sanción fraccionada para tres estrategias (un juego de bienes públicos opcional) se describe en el Capítulo E. Este trabajo fue publicado en la revista Games MDPI (Botta et al., 2021).

La discusión de los resultados obtenidos se presenta en el capítulo F, y las conclusiones en el capítulo G.

A.1 Objetivos

A.1.1 Objetivo general

- Definir y modelar la sanción o el castigo fraccionado en juegos de bienes públicos en el contexto de la teoría de juegos evolutivos y la evolución de la cooperación, analizando los resultados con énfasis en el incremento de la cooperación, los requisitos necesarios para alcanzar la cooperación total y señalando las semejanzas entre los resultados obtenidos y las observaciones empíricas de casos prácticos.

A.1.2 Objetivos específicos

- Definir el mecanismo de castigo fraccionado.
- Modelar el sistema en el contexto de la teoría de juegos evolutivos y la evolución de la cooperación.
- Presentar el castigo fraccionado para un juego de bienes públicos opcional y compulsivo con la dinámica del replicador.
- Analizar los puntos de equilibrio del sistema, su estabilidad.
- Presentar los requisitos necesarios para alcanzar la cooperación total.
- Señalar las coincidencias entre los resultados teóricos y la observaciones empíricas en casos prácticos.

A.2 Revisión bibliográfica

La cooperación es un comportamiento difícil de explicar. En la naturaleza abundan ejemplos en los cuales individuos de una misma especie cooperan entre sí; sin embargo, la propia existencia de la cooperación desafía a la lógica. La selección natural es un proceso competitivo que favorece a los individuos más aptos, es decir, individuos con rasgos y comportamientos que facilitan su supervivencia y, eventualmente, el paso de estas características favorables a la siguiente generación. Un individuo que coopera renuncia a parte de su propio beneficio para ayudar a otros; bajo estas condiciones, ¿cómo es posible que la cooperación exista y sobreviva? Por décadas, este acertijo ha fascinado a los biólogos evolutivos (Hamilton, 1964a; Haldane, 1955). Se encontró que ciertas circunstancias favorecen la cooperación (la selección de parentesco, la reciprocidad directa e indirecta, la reciprocidad en redes y la selección por grupo (Nowak, 2006)). Por su parte, en el área de la economía, se ha estudiado este problema a través de los dilemas sociales; en "La Tragedia de los Comunes" (Hardin, 1968), Hardin propone como solución la "Coersión mutua, mutuamente acordada". En forma complementaria Ostrom, en "Gobernando los Comunes" (Ostrom, 1990), define los principios encontrados en casos exitosos de manejo de recursos comunales. En ambas áreas de investigación, esta situación fue modelada usando la teoría de juegos (Trivers, 1971; Axelrod and Hamilton, 1981) y con el surgimiento de la teoría de juegos evolutivos (Smith, 1986), se modeló utilizando modelos de dinámica evolutiva (Taylor and Jonker, 1978; Weibull, 1997; Hofbauer and Sigmund, 1998).

En la literatura, dos mecanismos que afectan la cooperación fueron muy estudiados: la aplicación de incentivos y la posibilidad de abstenerse de participar. Los incentivos sean castigos o recompensas pueden aumentar la tasa de cooperación en un grupo (Yamagishi, 1986; Fehr and Gächter, 2002; Sigmund, 2007; Sasaki et al., 2012), en tanto que la posibilidad de rehusarse a participar es útil para sostener la cooperación en el tiempo (Hauert et al., 2002a,b).

Sin embargo, aplicar incentivos es costoso porque se utilizan recursos del sistema; por esa razón, el sistema de incentivos también es un bien público (Yamagishi, 1986). En la literatura, las dos formas de aplicar sanciones más utilizada en un juego de bienes públicos son el castigo por pares y el castigo por pozo (Sigmund et al., 2010b). Se diferencian entre sí, por el momento en que se decide si aplicar o no las sanciones y por la forma en que se obtiene el fondo que será utilizado para el sistema sancionador. En el castigo por pares, al finalizar el juego, cada individuo que aportó al juego (denominado como cooperador) decide si utiliza o no parte del beneficio que obtuvo para sancionar a aquellos que no aportaron (denominados como desertores). En cambio, en el castigo por pozo, cada cooperador antes de iniciar el juego aporta a un fondo que será utilizado exclusivamente para sancionar a los desertores. Ambos métodos fueron estudiados y comparados tanto en modelos teóricos como en experimentos (Fehr and Gächter, 2002; Sigmund et al., 2010a,b; Traulsen et al., 2012; Sasaki and Unemi, 2011) En los últimos años, en la literatura se presentaron otras formas de aplicar los incentivos inspirados en las condiciones encontradas en el mundo real. (Dercole et al., 2013) considera que la sanción es un monto fijo. En (Chen et al., 2014), un grupo de los cooperadores es seleccionado al azar para castigar a los desertores. En (Zhang et al., 2017a) el costo de sancionar es compartido entre todos los cooperadores y en (Zhang et al., 2017b, 2018), el pago máximo que puede obtener un desertor está restringido.

A diferencia de los trabajos mencionados anteriormente, en este modelo se asumen dos condiciones: i) que tanto el juego como las sanciones deben implementarse con el monto aportado periódicamente a un pozo común. Esta es una práctica común en instituciones o proyectos que proveen un servicio. Bajo esta circunstancia, un individuo que aporta para el servicio, implícitamente también está aportando al sistema de sanciones. ii) La institución decide la porción de los ingresos a ser utilizada para el servicio y para las sanciones. Así, el sistema de sanciones no representa un costo fijo, dependerá del número de infractores que existan pero también de la fracción de los mismos que se decida sancionar, esta fracción puede adaptarse a los recursos disponibles en la institución.

Capítulo B

Metodología y conceptos básicos

Considere una población grande de donde cada cierto tiempo se seleccionan n individuos al azar a los que se les ofrece participar en un juego de bienes públicos ($n = n_x + n_y + n_z$; $n \geq 2$). A los individuos que deciden no participar se les denomina solitarios (n_z). Los solitarios obtienen un pago constante σ que es independiente del resultado del juego. Los individuos que decidan participar del juego ($s = n_x + n_y$), deben decidir si contribuir o no contribuir al mismo con un monto $c = 1$ que será colocado en un pozo común. A los individuos que deciden contribuir se les denomina cooperadores (n_x) y a aquellos que no contribuyan se les denomina desertores (n_y). El monto total de las contribuciones se multiplica por un factor r ($1 < r < n$) y, en un juego de bienes públicos, el monto final se distribuye equitativamente entre todos los participantes del juego, sean ellos cooperadores o desertores. En cambio, en esta tesis, a un porcentaje previamente definido de desertores se les reducirá el pago a cero, mientras que los demás desertores recibirán el pago normal.

El juego, como ha sido definido anteriormente, consta de tres posibles estrategias y por lo tanto, un jugador puede pertenecer al grupo de: cooperadores x, desertores y o solitarios z. Los pagos obtenidos en un grupo de n_x cooperadores, n_y desertores, y n_z solitarios, considerando que cada contribución es igual ($c = 1$), se encuentran definidos de la siguiente forma (Hauert et al., 2002a, 2004):

$$p_x = r\frac{n_x}{s} - 1, \quad p_y = r\frac{n_x}{s}, \qquad p_z = \sigma, \tag{B.1}$$

donde los parámetros r, σ, y n cumplen las siguientes condiciones:

Axioma 1 *La tasa de interés o factor de multiplicación del pozo común r satisface $1 < r < n$.*

Si $1 < r$, i.e. si todos cooperan, el pago es mejor que si todos desertan. Si $r < n$, cada individuo recibe un mejor pago si deserta que si coopera (Hauert et al., 2002a).

Axioma 2 *El pago de la estrategia de los solitarios* σ *satisface* $0 < \sigma < r - 1$.

Así, el beneficio de un cooperador en un grupo de cooperadores (en donde cada uno obtiene $(r-1)$) es mejor que el beneficio de un solitario que obtiene σ, pero un solitario obtiene un mejor pago que un desertor en un grupo de desertores en donde el pago es igual a 0 (Hauert et al., 2002a).

En este trabajo, consideramos que una fracción d $(0 \leq d \leq 1)$ del grupo de desertores será sancionado, quedando su pago reducido a 0, mientras los demás desertores obtendrán el pago normal. Por ello, un desertor al desconocer si será castigado o no, en presencia de n_x cooperadores, tendrá un pago igual a:

$$p_y = (1-d)\left(\frac{rn_x}{s}\right) + d0. \tag{B.2}$$

Los pagos definidos en (B.1), son los pagos obtenidos cuando se conoce el número de individuos que decide jugar y el número de individuos que decide contribuir al juego, sin embargo, en un juego de bienes públicos un individuo desconoce la composición del grupo en el que jugará, es decir, ignora las estrategias de los demás participantes. Así, el pago esperado de un desertor en un grupo de n individuos, en donde el número de participantes s y el número de cooperadores n_x son variables aleatorias está definido de la siguiente forma

$$p_y(x,z,d) = \sigma z^{n-1} + (1-d)\, r\frac{x}{1-z}\left(1-a\right), \tag{B.3}$$

donde $a := (1 - z^n)/\left(n(1 - z)\right)$. El pago esperado de un cooperador en un grupo de n individuos seleccionados al azar se analiza de forma similar y está definido como

$$p_x(x,z) = \sigma z^{n-1} + ra + r\frac{x}{1-z}\left(1-a\right) + (1-r)z^{n-1} - 1. \tag{B.4}$$

La diferencia entre el pago de desertores y cooperadores denota la ventaja (o desventaja) de los desertores sobre los cooperadores, y es esencial para caracterizar la solución del sistema. Esta diferencia está dada por

$$g(x,z,d) := p_y - p_x = 1 + (r-1)z^{n-1} - ra - dr\frac{x}{(1-z)}\left(1-a\right). \tag{B.5}$$

Con el fin de resaltar el efecto del parámetro d, la expresión (B.5) se puede reescribir como

$$g(x,z,d) := m(z){-}dxh(z), \tag{B.6}$$

donde $m(z) = 1{+}(r{-}1)z^{n-1}{-}ra$ y $h(z) = \frac{r}{1-z}(1{-}a)$. Observe que, cuando $d = 0$, entonces $g(x,z,d) = m(z)$, y se recupera el sistema libre presentado en (Hauert et al., 2002a).

Para modelar la dinámica de la población, en este trabajo se sigue el modelo presentado en (Hofbauer and Sigmund, 1998). Cada individuo en la población utiliza una estrategia i (para $i = x, y, z$): cooperadores x, desertores y, y solitarios z. Sea $0 \leq x(t), y(t)$, y $z(t) \leq 1$ la frecuencia de cada una de las estrategias presentes en la población en un tiempo específico t. A fin de simplificar la notación, se obvia la dependencia del tiempo, así, la distribución de frecuencias de las estrategias en la población en un tiempo específico está definida por el estado $[x, y, z]$, que pertenece al simplex $\mathcal{S}_3$ dado por:

$$\mathcal{S}_3 = \left\{[x,y,z] \in \mathbb{R}^3 : x,y,z \geq 0 \text{ y } x+y+z=1\right\}. \tag{B.7}$$

El interior del simplex $\mathcal{S}_3$ está definido como el conjunto de puntos en donde todas las estrategias están presentes (*i.e.* x > 0, y > 0, y z > 0) y el borde de $\mathcal{S}_3$ como el conjunto de puntos que no son puntos interiores. El borde del simplex comprende vértices y bordes. Los vértices x, y, y z corresponde a puntos del simplex en donde la población es homogénea; una población compuesta exclusivamente por cooperadores, desertores y solitarios respectivamente. Los bordes xy, yz, y zx, corresponde a puntos en donde una de las estrategias está ausente.

Asumiendo una población suficientemente grande, con generaciones que se combinan continuamente, cada estrategia evoluciona, siguiendo la ecuación del replicador, de acuerdo a la tasa que define su éxito evolutivo en la población. El sistema de las tres estrategias se define de la siguiente forma:

$$\begin{cases} \dot{x} = x\left(p_x{-}\bar{p}\right) \\ \dot{y} = y\left(p_y{-}\bar{p}\right) \\ \dot{z} = z\left(p_z{-}\bar{p}\right) \end{cases} \tag{B.8}$$

donde $x(0)$, $y(0)$, y $z(0)$ son las condiciones inciales en $\mathcal{S}_3$, p_i es el pago de la estrategia i para $i = x, y, z$ y, $\bar{p} = xp_x{+}yp_y{+}zp_z$, es el pago promedio de la población. En la Expresión (B.8), se puede ver cómo la frecuencia de la estrategia i, depende de la diferencia entre el pago p_i de la misma y el pago promedio de la población $\bar{p}$. Cuando

$p_i > \bar{p}$, la frecuencia de la estrategia i aumenta, en caso contrario, disminuye.

Para simplificar el análisis en el interior del $\mathcal{S}_3$ se define una variable $f := x/(x+y)$ (Hauert et al., 2002a) que representa la fracción de cooperadores entre los participantes del juego y permite definir la biyección $\mathcal{T} : (x,y,z) \rightarrow (f,z)$. Adicionalmente, la restricción $x+y+z=1$ debe cumplirse. Así, el sistema (B.8) puede reescribirse en forma reducida con dos variables (Hauert et al., 2002a):

$$\begin{cases} \dot{f} &= -f(1-f)g(f,z,d) \\ \dot{z} &= z(\sigma - \bar{p}(f,z)) \end{cases} \tag{B.9}$$

donde $(f,z) \in [0,1] \times [0,1]$ y $g(f,z,d) = 1 + (r-1)z^{n-1} - ra - drf\,(1-a)$. A partir de este sistema se puede obtener el valor de f:

$$\hat{f} = \frac{\sigma}{(1-d)r} \frac{(1-z^{n-1})}{(1-a)}. \tag{B.10}$$

Con el valor de f, se obtiene una definición equivalente para $g(\cdot)$ que depende sólamente de z y d:

$$\tilde{g}(z,d) = m(z) + \tilde{h}(z,d), \tag{B.11}$$

donde $\tilde{h}(z,d) = -\frac{d}{1-d}\sigma(1-z^{n-1})$ y $m(z) = 1 + (r-1)z^{n-1} - ra$ (como fuera definida en (B.6)). A partir de (B.11), el valor de z en el equilibrio, definido como $\hat{z}$, se puede obtener numéricamente como una función de d. Asimismo, se obtiene una nueva forma de $\hat{f}$ que depende únicamente de d y está definido por:

$$\hat{f} = \frac{\sigma}{(r-1) + d(\sigma - (r-1))}, \tag{B.12}$$

así, el punto de equilibrio interior $(\hat{f}, \hat{z})$, se corresponde en el sistema $(\hat{x}, \hat{y}, \hat{z})$ al punto de equilibrio interior del simplex $\mathcal{S}_3$ dado por

$$(\hat{x}, \hat{y}, \hat{z}) = \left(\frac{\sigma}{\alpha}(1-\hat{z}), (1-\frac{\sigma}{\alpha})(1-\hat{z}), \hat{z} \right)$$

donde $\alpha = (r-1) + d(\sigma - (r-1))$ y la dependencia del parámetro d es dado explícitamente. En la Tabla B.1 se presenta un resumen de los puntos de equilibrio en el sistema $(\hat{x}, \hat{y}, \hat{z})$ y su correspondencia en el sistema $(\hat{f}, \hat{z})$.

Tabla B.1: Equivalencia de los puntos de equilibrio entre los sistemas $(\hat{x}, \hat{y}, \hat{z})$ y $(\hat{f}, \hat{z})$. $\alpha = (r-1) + d(\sigma - (r-1))$, $\beta = (n-r)/(r(n-1)d)$.

$(\hat{x}, \hat{y}, \hat{z})$	$(\hat{f}, \hat{z})$
$(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$	$(\hat{f}, \hat{z}) = (1, 0)$
$(\hat{x}, \hat{y}, \hat{z}) = (0, 1, 0)$	$(\hat{f}, \hat{z}) = (0, 0)$
$(\hat{x}, \hat{y}, \hat{z}) = (0, 0, 1)$	$(\hat{f}, \hat{z}) = (0, 1)$
$(\hat{x}, \hat{y}, \hat{z}) = (\frac{\sigma}{\alpha}(1-\hat{z}), (1-\frac{\sigma}{\alpha})(1-\hat{z}), \hat{z})$	$(\hat{f}, \hat{z}) = (\frac{\sigma}{\alpha}, \hat{z})$
$(\hat{x} = f, \hat{y} = 1 - f, \hat{z} = 0)$	$(\hat{f}, \hat{z}) = (\beta, 0)$

Los primeros tres puntos de equilibrio en la Tabla B.1 corresponden a los vértices del simplex $\mathcal{S}_3$, los otros dos son puntos de equilibrio interiores: el cuarto en el interior de $\mathcal{S}_3$ y el quinto en el interior del borde xy. Con el fin de distinguir el efecto de d sobre los puntos de equilibrios del sistema $(\hat{f}, \hat{z})$, a partir de ahora se define como $(\hat{f}_d, \hat{z}_d)$ al punto de equilibrio cuando $d > 0$ y se define como $(\hat{f}_0, \hat{z}_0)$ al punto de equilibrio cuando $d = 0$.

Capítulo C

Antecedentes del trabajo

En una etapa anterior a este trabajo, se realizó una revisión bibliográfica de modelos que estudiaran la evolución de la cooperación en un juego de bienes públicos utilizando la dinámica del replicador. Algunos de los modelos fueron sistematizados en una tabla (ver Tabla C.1), de acuerdo a sus características, con el fin de evaluar cuáles podrían ser aplicados a casos prácticos.

Tabla C.1: Tabla comparativa. Modelos de juegos de bienes públicos

Modelo	Año	Referencia	Tipo de Juego[a]	Estrategias [b]	Tipo de incentivo [c]	Castigo de segundo orden	Población	Condiciones
1	2002	Hauert et al.	OPGG	C, D, S	-	-	Grande	$r > 1$, $0 < \sigma < r - 1$
2	2005	Fowler	OPGG	C, D, S, P	Peer P	Si	Grande	$r - 1 > \sigma$, $p > c$
3	2006	Brandt et al.	OPGG	C, D, S, P	Peer P	Si	Grande	$n > r > 1 + \sigma$, $\beta > 1 > \alpha > 0$
4	2008	Hauert et al.	OPGG	C, D, S, P	Peer P	Si	Infinita Finita	$r < n$ $0 \leq \alpha < 1$, $(r-1)c > \sigma > 0$, $\beta > \gamma$
5	2011	Sasaki and Unemi	CPGG	C, D, R	Pool R	Si	Infinita	$n \geq 2$ $r_1 < n$
6	2012	Sasaki et al.	OPGG	C, D, S	Pool R Pool P	-	Grande	$n \geq 2$,$m \geq 2$, $1 < r < n$

Nota. Esta tabla proviene del trabajo (Botta et al., 2014), en donde se puede obtener la descripción completa. A continuación se definen los conceptos más importantes.

[a] OPGG: Juego de bienes públicos opcional (también conocido como Juego de bienes públicos voluntario -VPGG-), CPGG: Juego de bienes públicos compulsorio

[b] C: Cooperadores, D: Desertores, S: Solitarios, P: Castigador, R: Recompensador

[c] PeerP: Castigo por pares, PoolP: Castigo por pozo, PoolR: Recompensa por pozo

Los modelos seleccionados variaban en cuanto a número de estrategias, tipo y forma de aplicar los incentivos, y tamaño de la población. Tres de estos modelos fueron estudiados a profundidad; se implementaron y probaron con datos generales provenientes de JS del país (ABC, 2012; Martínez et al., 2012) (ver Figura C.1). Las JS son organizaciones comunitarias con características particulares en cuanto a su creación y administración. El gobierno asiste a las juntas durante su proceso de consolidación, sin embargo, una vez constituidas, la administración es responsabilidad de la comunidad. Por décadas, estas organizaciones han sido clave para la obtención de agua potable en zonas suburbanas y rurales del país. Los resultados obtenidos de este trabajo, así como algunas sugerencias para los tomadores de decisiones de las JS se encuentran en (Botta, 2013; Botta et al., 2014). A través del proyecto de investigación (Botta et al., 2017a), se buscó recabar datos de morosidad en las JS para comparar los datos de campo con el resultado teórico y la información obtenida de los medios de comunicación. El resultado fue poco satisfactorios en cuanto a la cantidad de información obtenida: esto se debió a que los datos a menudo son considerados como información sensible sobre el desempeño de una junta. Aún así, el proyecto resultó sumamente útil para entender el modus operandi de las mismas.

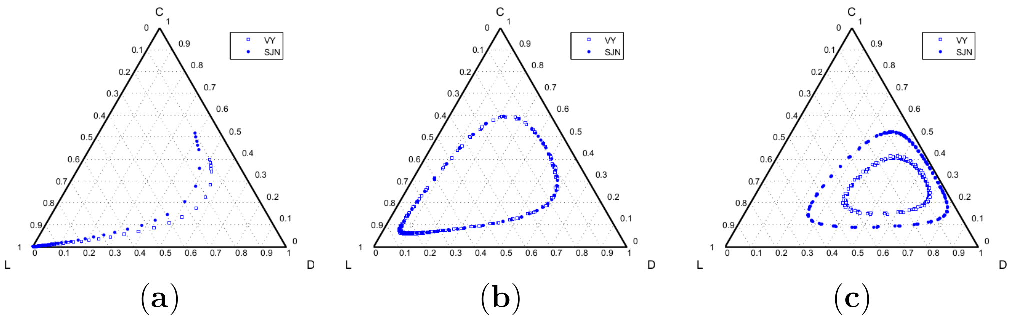
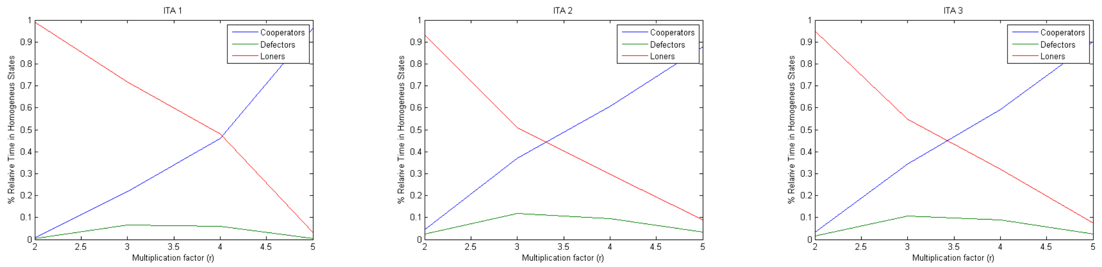


Figura C.1: Juntas de Saneamiento. Modelo 1. Juego de bienes públicos opcional sin incentivos. Cuando el juego es voluntario, aún sin incentivos es posible sostener la cooperación en el tiempo cuando $r > 2$ Parámetros: $n = 5$, $\sigma = 1$, (**a**) $r = 2$, (**b**) $r = 3$ y (**c**) $r = 3,5$. Condiciones iniciales: San Juan Nepomuceno (SJN) $x_c = 0,52$; $x_d = 0,38$; $x_r = 0,1$ y Villa Ygatimi (VY) $x_c = 0,4$; $x_d = 0,5$; $x_r = 0,1$. *Nota.* Esta figura proviene del trabajo (Botta et al., 2014), en donde se puede obtener la descripción completa.

En paralelo, se realizaron trabajos puntuales relacionados con modelos centrados en poblaciones finitas y control (Botta et al., 2016, 2018, 2017b). En (Botta et al., 2016) se estudió el modelo de (Hauert et al., 2007) para analizar la distribución estacionaria de un sistema y la probabilidad de pasar de un estado a otro (cooperador, desertor, solitario y castigador) de acuerdo al tamaño de la población; el modelo fue implementado en (Botta et al., 2018) con los datos del proyecto de investigación (Botta et al., 2017a)(ver Figura C.2).

De acuerdo a lo observado en estos trabajos, se pudo concluir, que los modelos estudiados son adecuados para experimentos realizados en laboratorio pero son difíciles de aplicar a un caso práctico, como por ejemplo los proyectos de grupos comunitarios, porque difieren tanto en los procedimientos como en las características. Por lo tanto, se elige un modelo básico de juego de bienes públicos opcional sin incentivos (ver Tabla C.1 Modelo 1) y se agrega un modelo de incentivo comúnmente utilizado en la práctica, sancionar únicamente a una fracción de los infractores. En



(**a**) (**b**) (**c**)

Figura C.2: Tiempo relativo en estados homogéneos como una función del factor de multiplicación r para JS de diferente tamaño. Al aumentar r se aumenta el tiempo que el sistema pasa en el estado de cooperación. El tiempo depende ademas del tamaño de la población (M). Para JS de mayor tamaño, r debe ser mayor: Para alcanzar al menos 50% del tiempo en el estado de cooperación, ITA1 ($M = 1123$) necesita $r \geq 4$, pero para ITA2 ($M = 141$) e ITA3 ($M = 205$), con menos población, $r \geq 3$ is suficiente. Parámetros: $N = 5$, $\sigma = 1$, $c = 1$, $r = 3$, $s = 0.249$, $b = 1$, $g = 0.3$; (**a**) $M = 1123$; (**b**) $M = 141$ y (**c**) $M = 205$. *Nota*. Esta figura proviene del trabajo (Botta et al., 2016), en donde se puede obtener la descripción completa.

el siguiente capítulo se presenta el castigo fraccionado aplicado a un juego de bienes públicos compulsorio.

Capítulo D

Castigo fraccionado para dos estrategias

El modelo de sancionamiento fraccionado presentado en esta tesis, fue implementado primeramente en un modelo de juego de bienes públicos compulsorio con dos estrategias, cooperar o desertar. Como en este juego no existe la opción de no participar, la estrategia de los solitarios desaparece, *i.e.* $z = 0$, y los pagos presentados en la metodología (Ecuaciones B.3, B.4 y B.5) se reducen a

$$\begin{aligned} p_y =& (1-d) r \left(1-\frac{1}{n}\right) x \\ p_x =& \frac{r}{n}((n-1)x+1)-1 \\ p_y - p_x =& \left(1-\frac{r}{n}\right) - d\left(r-\frac{r}{n}\right) x =: g(d,x), \end{aligned} \tag{D.1}$$

y el sistema B.8 se reduce a

$$\begin{cases} \dot{x} = x\,(p_x - \bar{p}) \\ \dot{y} = y\,(p_y - \bar{p}) \end{cases} \tag{D.2}$$

En el borde xy, $x+y=1$, y la dinámica puede ser analizada con la siguiente ecuación:

$$\dot{x} = -x(1-x)g(x,d) = -x(1-x)(1-(r/n)-drx(1-1/n)), \tag{D.3}$$

que tiene los siguientes puntos de equilibrio: $\hat{x}=1$ ($\hat{y}=0$), $\hat{x}=0$ ($\hat{y}=1$) y $\hat{x}=(n-r)/(r(n-1)d)$. Estos puntos de equilibrio fueron caracterizados en (Botta et al., 2020) usando el siguiente lema:

Lemma 4 *Considere la Ecuación (D.3) que modela el borde xy donde x denota a los cooperadores, y denota a los desertores, $x \geq 0$, $y \geq 0$, y $x+y=1$. Si la fracción de desertores sancionada se denota con d, siendo $0 \leq d \leq 1$ y definiendo $d_1 := (n-r)/(r(n-1))$, entonces (1) para $0 \leq d \leq 1$, el punto de equilibrio $\hat{x}=0$ es localmente asimptóticamente estable, (2) $\hat{x}=1$ es localmente asimptóticamente estable para $d_1 < d \leq 1$, y (3) el punto $\hat{x}=(n-r)/(r(n-1)d)$, que existe en el interior del borde para $d_1 < d$, es un punto de equilibrio inestable.*

Los resultados posibles del sistema presentado en el Lema 3.1 están esquematizados en la Figura D.1. Cabe mencionar la dependencia de los puntos de equilibrio de la función $g(x,d)$. Si $0 \leq d < d_1$, entonces $g(x,d) \neq 0$ para todos los valores de x, y los dos puntos de equilibrio son $\hat{x}=0$ y $\hat{x}=1$ son globalmente asimptóticamente estable y globlamente asimptóticamente inestable respectivamente (ver la Figura D.1.(a)). Si $d_1 < d$, un punto de equilibrio interior inestable $\tilde{x}$ aparece en el interior del borde (ver Figura D.1.(b)). Si $d=(n-r)/r(n-1)x$, entonces $g(x,d)=0$, y el estado x corresponde al punto de equilibrio $\tilde{x}$.

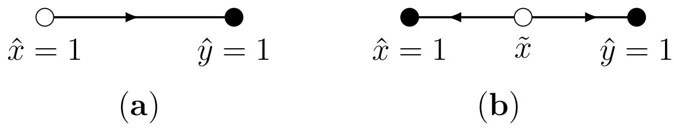


Figura D.1: Estabilidad de los puntos de equilibrio del sistema para diferentes valores de d: el punto negro representa un equilibrio estable y el punto blanco a un equilibrio inestable. (**a**) Si $d < d_1$, $\hat{x} = 1$ es inestable y $\hat{y} = 1$ es asimptóticamente estable. (**b**) Si $d_1 < d \leq 1$, un punto de equilibrio inestable $\hat{x} = (n - r)/r(n - 1)d$, definido como $\tilde{x}$, aparece, cambiando la estabilidad de $\hat{x} = 1$, que se vuelve un punto de equilibrio estable.

La función $g(x, d)$ representa la diferencia entre los pagos de cooperadores y desertores; su valor indica cuál estrategia es favorecida, y eventualmente, el estado final del sistema. El valor de $g(x, d)$ depende de los parámetros r, n y d, pero también depende de la frecuencia de cooperadores en la población (ver Figura D.2).

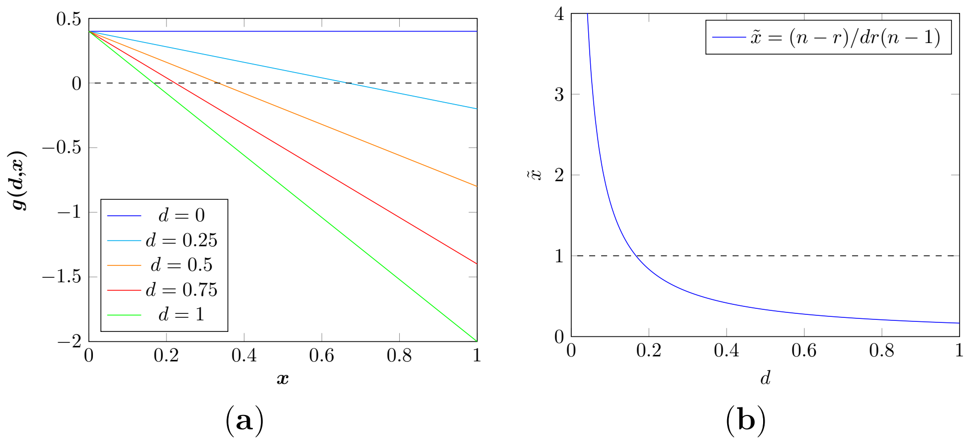


Figura D.2: (**a**) $g(x, d)$ en función de x para diferentes valores de d: cuando $d = 0$, $g(x, d)$ es positivo para todos los valores de x. Si $d > 0$, $g(x, d)$ cambia de signo cuando $\hat{x} = (n - r)/(r(n - 1)d)$. (**b**) Relación entre $\hat{x}$ y d, cuando d aumenta, $\hat{x}$ disminuye. Como $0 \leq \hat{x} \leq 1$, $\hat{x} \geq 1$ no tiene sentido en el sistema, o visto de otra forma, $d \leq (n - r)/(r(n - 1)\hat{x})$ no produce efecto sobre el sistema. Parámetros: $r = 3$, $n = 5$.

De acuerdo a los resultados obtenidos, sancionando solamente a una fracción de los desertores es posible mejorar el porcentaje de cooperación. El resultado depende de la frecuencia inicial de cooperadores (x). En particular, si la frecuencia inicial es menor a un umbral, sancionar, inclusive a todos los desertores no produce el efecto deseado y el estado final del sistema es una población compuesta solamente de desertores. Si la frecuencia inicial de cooperadores (x) en la población sobrepasa el umbral, sancionando a los desertores se puede obtener una cooperación total, que en términos prácticos significa que todos los participantes cooperan y el proyecto o institución es exitoso. El valor de d necesario para alcanzar la cooperación total dependerá de los parámetros r, n y de x; cuanto mayor es la frecuencia inicial de x, menor es el valor de d necesario para alcanzar la cooperación total.

Sancionar solamente a un grupo de los desertores reduce el costo de sancionar a los que no cumplen con las normas y es una solución pragmática cuando los recursos no son suficientes. En el siguiente capítulo, el modelo se extiende a tres estrategias, modelando un juego de bienes públicos opcional.

Capítulo E

Castigo fraccionado para tres estrategias

Para analizar el juego de bienes públicos opcional utilizaremos los pagos de las estrategias presentados en el Capítulo B. En la primera parte, se analizará el borde del simplex, en particular los bordes xz y zy. Se demostrará el efecto del parámetro d sobre los valores de f y z en el equilibrio, luego se analizarán cualitativamente los puntos de equilibrio y se estudiará la cuenca de atracción del punto de equilibrio $x = 1$.

E.1 Borde del simplex

En el borde, el sistema (B.8) se reduce a dos estrategias. En el Capítulo D se analizó el borde xy, a continuación se presenta el análisis de los bordes zx e yz. Se observa cómo el parámetro d no tiene efecto sobre el comportamiento del sistema en los bordes zx e yz. Ademas, como en el juego se define $0 < \sigma < r - 1$, la dinámica del borde yz va del vértice y al vértice z y la dinámica del borde xz va del vértice z al vértice x.

Borde zx. Este borde representa a una población de cooperadores y solitarios. Como $x + z = 1$ $(y = 0)$, es suficiente con analizar la ecuación $\dot{x} = ((r-1) - \sigma)x(1 - x)\,(1 - z^{n-1})$ para caracterizar el sistema. Como por el juego $\sigma < (r - 1)$, los puntos de equilibrio son: $\hat{x} = 1$ $(\hat{z} = 0)$ y $\hat{x} = 0$ $(\hat{z} = 1)$. La Jacobiana de $\dot{x}$ presenta la siguiente forma:

$$J(x,z) = (1-2x)\Big(((r-1)-\sigma)(1-z^{n-1})\Big) \tag{E.1}$$

Evaluando (E.1) se puede obsevar que el punto de equilibrio $\hat{x} = 1$ ($\hat{x} = 0$) es estable (inestable) cuando $(r-1) > \sigma$, como lo requiere el juego y la dinámica del borde zx va del vértice z a vértice x

Borde yz. Este borde representa una población compuesta de desertores y solitarios. Como $y + z = 1$ ($x = 0$), el sistema se puede analizar completamente por la ecuación: $\dot{y} = y(1-y)\sigma(z^{n-1} - 1)$. Como $0 < \sigma$ por los requisitos del juego, los puntos de equilibrio son $\hat{y} = 1$ ($\hat{z} = 0$) y$\hat{y} = 0$ ($\hat{z} = 1$). La Jacobiana de $\dot{y}$ presenta la siguiente forma:

$$J(y,z) = (1-2y)\sigma(z^{n-1} - 1) \tag{E.2}$$

Como $0 < \sigma$ el punto de equilibrio $\hat{y} = 1$ ($\hat{y} = 0$) es inestable (estable) y la dinámica del borde yz va del vértice y a vértice z.

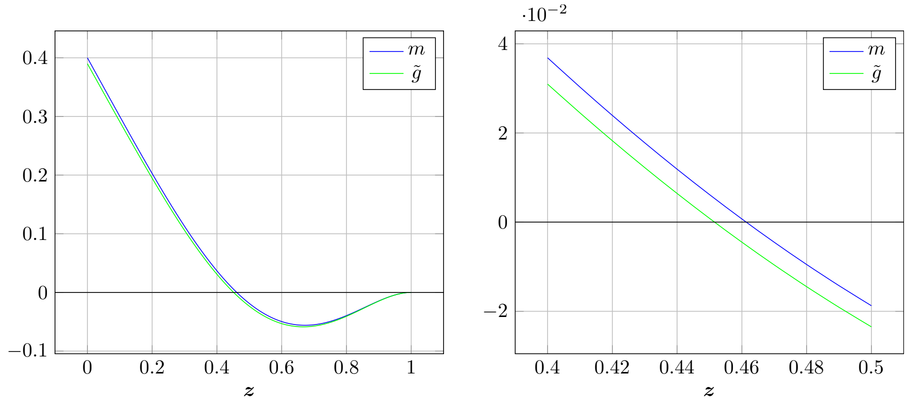
E.2 Interior del simplex

Un punto en el interior del simplex representa una población en la que todas las estrategias están presentes: cooperadores, desertores y solitarios. El parámetro d, modifica la posición, la naturaleza y el número de puntos de equilibrios del sistema. Dos valores de d son especialmente importantes en el análisis del trabajo, el primero es $d_1 := (n-r)/(r(n-1))$, definido en el Lema 4, y que corresponde al valor de d cuando $x = 1$. El segundo es $d_2 = (n-r)/(n\sigma + n - r)$, el cual proviene de la reducción del la Expresión (B.11) cuando $z = 0$. Ambos valores son utilizados en las siguientes secciones.

Efecto de d sobre el valor de z en el equilibrio. Para analizar el efecto del parámetro d en $\hat{z}$ y en las funciones $g(f,z,d)$ y $\tilde{g}(z,d)$, considere que, para $d_0 = 0$, existe un valor de $\hat{z}_0 \in (0,1)$ tal que $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0) = 0$ (ver Expresiones (B.11) y (B.6)). Se ha encontrado que cuando $d > d_0$, el valor de $\hat{z}_0$ se desplaza a otro valor $\hat{z}_d$ tal que $\hat{z}_d < \hat{z}_0$. El efecto del parámetro d en la solución $\hat{z}_d$ con $\tilde{g}(z,d) = 0$ con respecto a la solución $\hat{z}_0$ con $\tilde{g}(\hat{z}_0, d_0) = m(\hat{z}_0) = 0$ se analiza en el siguiente lema.

Lemma 5 *Considerando los valores de $\hat{z}_0$ y $\hat{z}_d$ asociados a $d_0 = 0$ y $d > d_0$ tal que $\tilde{g}(\hat{z}_0, 0) = m(\hat{z}_0) = 0$ y $\tilde{g}(\hat{z}_d, d) = 0$; entonces, $\hat{z}_d < \hat{z}_0$.*

En la Figura E.1 se puede observar el efecto de d al desplazar $\hat{z}$ hacia la izquierda; *i.e.*, $\hat{z}_d$ con $d > 0$ es menor que $\hat{z}$ con $d = 0$.



(**a**) (**b**)

Figura E.1: Modelo para tres estrategias. Gráfico de (**a**) $m(z)$ y $\tilde{g}(z,d)$ en el intérvalo $z \in (0,1)$ con $d = 0.01$. (**b**) En la cercanía de $m(z) = 0$ y $\tilde{g}(z,d) = 0$, puede observarse mejor como d disminuye el valor de $\hat{z}$ en el equilibrio y desplaza el equilibrio $\hat{z}$ hacia el borde xy donde $z = 0$. Parámetros: $n = 5$, $r = 3$ y $\sigma = 1$.

Efecto de d sobre el valor de f en el equilibrio. En la metodología, se definió que en el punto de equilibrio interior $\hat{f}_d = \sigma/((r-1) + d(\sigma - (r-1)))$, además, es un requerimiento del juego que $\sigma < (r-1)$ (Hauert et al., 2002a) y como consecuencia se tiene la siguiente expresión $(r-1) + d(\sigma - (r-1)) < (r-1)$ para todo $0 < d < 1$. Así, para una $0 \leq d < \tilde{d}$ y los correspondientes $\hat{f}_0$, $\hat{f}_d$ y $\hat{f}_{\tilde{d}}$, se tiene la siguiente relación :

$$\hat{f}_0 = \frac{\sigma}{(r-1)} \leq \hat{f}_d = \frac{\sigma}{(r-1) + d(\sigma - (r-1))} < \hat{f}_{\tilde{d}} = \frac{\sigma}{(r-1) + \tilde{d}(\sigma - (r-1))}. \quad \text{(E.3)}$$

Por lo tanto, el parámetro d incrementa el valor de $\hat{f}$ en el equilibrio, *i.e* , aumenta el nivel de cooperación en el juego.

Efecto de d en el punto de equilibrio interior. Usando la definición de a (introducida en (B.3)), la Ecuación (B.9) puede ser reescrita como sigue:

$$\begin{cases} \dot{f} &= -f(1-f)\, g(f,z,d) \\ \dot{z} &= z(1-z)\Big(\Big(1 - z^{n-1}\Big)\Big(\sigma - f(r-1)\Big) - dfr\,(1-a)\,(f-1)\Big). \end{cases} \quad \text{(E.4)}$$

Para simplificar la lectura, los argumentos de $g(f,z,d)$ se obviarán y se escribirá solamente g. Para caracterizar el equilibrio, se observa la Jacobiana del Sistema (E.4), que presenta la siguiente forma:

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22}, \end{pmatrix}$$

donde

$$J_{11} = d\dot{f}/df = -((1-2f)g - f(1-f)dr(1-n)), \tag{E.5}$$

$$J_{12} = d\dot{f}/dz = -f(1-f)((n-1)(r-1)z^{n-2} - rn' + drfn'), \tag{E.6}$$

$$J_{21} = d\dot{z}/df = z(1-z)((1-r)(1-z^{n-1}) - (1-a)dr(2f-1)) \tag{E.7}$$

$$J_{22} = d\dot{z}/dz = (1-2z)((1-z^{n-1})b - e(1-a)) + z(1-z)(-(n-1)z^{n-2}b - e(-a')) \tag{E.8}$$

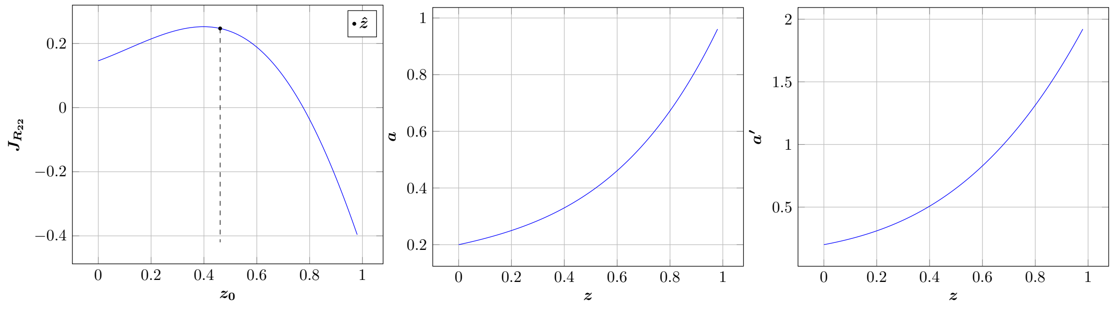
siendo $a' = (1 - z^n - nz^{n-1}(1-z)) \, / \, (n(1-z)^2)$, $b = \sigma - f(r-1)$ y $e = drf(f-1)$.

En el punto de equilibrio interior, $g = 0$ y $(1 - z^{n-1})b - e(1-a) = 0$, y, al evaluar la Jacobiana se obtiene

$$J = \begin{pmatrix} f(1-f)dr(1-a) & -f(1-f)\Big((n-1)(r-1)z^{n-2} \\ & -ra' + drfa'\Big) \\ & \\ z(1-z)\Big((1-r)(1-z^{n-1}) & z(1-z)\Big(-(n-1)z^{n-2}b \\ -(1-a)(2drf - dr)\Big) & -e(-a')\Big) \end{pmatrix}. \tag{E.9}$$

Para simplificar el análisis, considerando la influencia del parámetro d, la Jacobiana se separa en dos matrices J_R y J_T de dimensión 2×2 como $J = J_R + d\, J_T$.

Si $d = 0$, la matriz se reduce a J_R y al modelo en (Hauert et al., 2002a). Cuando es evaluada en $(\hat{f}_0, \hat{z}_0)$, los términos en la diagonal $J_{R_{11}}$ y $J_{R_{22}}$ son iguales a cero. El término $J_{R_{21}}$ es negativo porque $r > 1$. Además, el término $J_{R_{12}}$ es positivo para $\hat{z}_0 \in (0, 1)$ y los parámetros n, σ, y r con valores normalmente utilizados en la literatura. En la Figura E.2 se puede observar el valor de $J_{R_{12}}$ para $0 \leq z_0 \leq 1$ y en particular, el valor en el equilibrio $(\hat{f}_0, \hat{z}_0)$. Por lo tanto, los autovalores de J_R dados por $\lambda^2 = J_{R_{21}} J_{R_{12}}$, son imaginarios puros y el punto interior es un centro como fuera probado en (Hauert et al., 2002a).



(a) (b) (c)

Figura E.2: Modelo para tres estrategias. Valores de $J_{R_{12}}$, a y a' para valores crecientes de z. (**a**) Valores de $J_{R_{12}}$ para $0 \leq z_0 \leq 1$ y $\hat{f}_0 = \sigma/(r-1)$; para $\hat{z}_0$, el término es positivo. (**b**) Valores de a para $0 \leq z < 1$. (**c**) Valores de a' para $0 \leq z < 1$. Parámetros: $n = 5$, $r = 3$, y $\sigma = 1$.

En el siguiente Teorema se analiza la matriz Jacobiana J perturbada por el parámetro d. Para esto, se considera que $d \approx 0$.

Teorema 1 *Suponga una matriz Jacobiana del Sistema (6.11) definida como J y descompuesta como $J = J_R + dJ_T$ con entradas $J_{R_{ij}}$ y $J_{T_{ij}}$ para $i, j \in \{1, 2\}$ con derivadas acotadas. Suponga también un punto de equilibrio inicial $(\hat{f}_0, \hat{z}_0)$ del Sistema (E.4) que corresponde al parámetro $d = 0$, y otro punto de equilibrio para $0 < d$ definido como $(\hat{f}_d, \hat{z}_d)$. Entonces, la Jacobiana J, evaluada en $(\hat{f}_d, \hat{z}_d)$ con $d \approx 0$, tiene autovalores complejos con parte real positiva.*

El Teorema 1 muestra que para un valor pequeño de d, los autovalores de la Jacobiana J son complejos con parte real positiva cuando es evaluada en $(\hat{f}_d, \hat{z}_d)$. Esto significa que el punto de equilibrio es un foco inestable; *i.e.*, las soluciones del Sistema (E.4) son repelidos del equilibrio hacia el borde del simplex. En la Tabla E.1, se puede observar el punto de equilibrio $(\hat{f}_d, \hat{z}_d)$, las entradas de la Jacobiana J y los autovalores para algunos valores de d.

Cuando $d = 0$, los autovalores de J son imaginarios puros y el punto de equilibrio es un centro; si el valor de d es positivo, los autovalores tienen parte real positiva y el punto de equilibrio es un foco inestable, en concordancia con el Teorema 1. Asimismo, a medida que el valor de d se incrementa, valor de $\hat{z}_d$ decrece mientras que el valor de $\hat{f}_d$ se incrementa. Además, el punto de equilibrio $(\hat{f}_d, \hat{z}_d)$ cambia su posición desde $(\hat{f}_0, \hat{z}_0)$ con $d = 0$ en el interior del simplex hacia el borde xy (ver Figura E.3).

Tabla E.1: Efecto del parámetro d en las entradas de la jacobiana J evaluada en el punto de equilibrio $(\hat{f}_d, \hat{z}_d)$ con sus correspondientes autovalores. Para $d = 0$, $(\hat{f}_d, \hat{z}_d) = (\hat{f}_0, \hat{z}_0)$. Parámetros: $n = 5$, $r = 3$, y $\sigma = 1$; $\psi = J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11}$ y $\Delta = (J_{R_{22}} - \varepsilon_{22} + \varepsilon_{11})^2 - 4\,(J_{R_{21}} \pm \varepsilon_{21})(J_{R_{12}} + \varepsilon_{12})$.

d	$J = J_R + d\,J_T$	$(\hat{f}, \hat{z})$	$\lambda = 0.5\,(\psi) \pm 0.5\,\sqrt{\Delta}$
$d = 0.01$	$\begin{bmatrix} 0 & 0.24831 \\ -0.47471 & 0.00046 \end{bmatrix} + d \begin{bmatrix} 0.48162 & -0.21732 \\ -0.00437 & -0.06878 \end{bmatrix}$	$(0.5025, 0.4516)$	$\lambda_1 = 0.00210 - 0.34182i$ $\lambda_2 = 0.00210 + 0.34182i$
$d = 0.001$	$\begin{bmatrix} 0 & 0.24688 \\ -0.47454 & 0.00005 \end{bmatrix} + d \begin{bmatrix} 0.47778 & -0.22112 \\ -0.00044 & -0.06996 \end{bmatrix}$	$(0.5003, 0.4604)$	$\lambda_1 = 0.00021 - 0.34213i$ $\lambda_2 = 0.00021 + 0.34213i$
$d = 0$	$\begin{bmatrix} 0 & 0.24671 \\ -0.47450 & 0 \end{bmatrix} + d \begin{bmatrix} 0.47735 & -0.22155 \\ 0 & -0.07008 \end{bmatrix}$	$(0.5, 0.4613)$	$\lambda_1 = -0.34215i$ $\lambda_2 = +0.34215i$

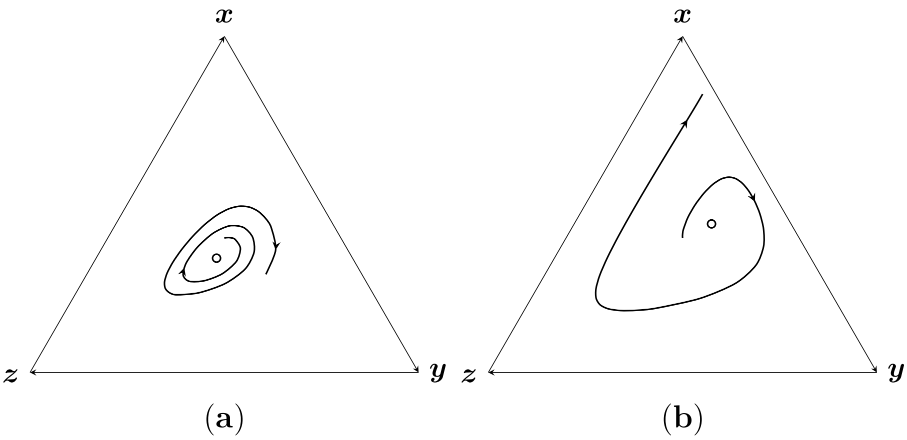


Figura E.3: Efecto de d en el punto de equilibrio interior. Parámetros: $n = 5$, $r = 3$, y $\sigma = 1$, y valores de d: (**a**) $d = 0.1$ y (**b**) $d = 0.2$. El punto blanco representa un punto de equilibrio inestable.

E.3 El punto de equilibrio $(\hat{x}, \hat{y}, \hat{z}) = (1, 0, 0)$

Este punto corresponde al vértice $x = 1$ y representa a una población homogénea compuesta exclusivamente por cooperadores. Cuando $d = 0$, este es un punto de equilibrio tipo silla [1] (Hauert et al., 2002b). A continuación, se caracteriza el equilibrio para $0 < d$ y en particular para $d_1 < d$. Por simplicidad, se utiliza el Sistema (f, z) (ver (B.9)). Evaluando la Jacobiana en este punto de equilibrio se obtiene:

$$J = \begin{bmatrix} 1 - r/n - dr\,(1 - 1/n) & 0 \\ 0 & \sigma - (r - 1) \end{bmatrix}. \tag{E.10}$$

[1] Los puntos de equilibrio tipo silla, actua como un atractor para algunas trayectorias y un repulsor para otras. En este trabajo se considera a los puntos tipo silla como puntos de equilibrio inestables

Los autovalores de (E.10) son: $\lambda_1 = 1 - r/n - dr\,(1 - 1/n)$ y $\lambda_2 = (\sigma - (r - 1))$. Note que λ_2 es negativo por la condición del juego de que $0 < \sigma < (r - 1)$ (Axioma 2) y la elección del valor de d determina el signo de λ_1. Dado que $d_1 = (n - r)/r(n - 1)$. Entonces, para $d < d_1$, λ_1 es positivo y el equilibrio es inestable (una silla), sin embargo, si $d_1 < d$, λ_1 es negativo y el equilibrio es localmente asimptóticamente estable. Para una mejor comprensión de la cuenca de atracción de este equilibrio, se considera la función de entropía relativa utilizada en (Weibull, 1997; Sandholm and Ansell, 2010) como función candidata de Lyapunov.

Teorema 2 *Suponga que* $(\hat{f}, \hat{z}) = (1, 0)$ *y la función candidata de Lyapunov* $V_{(\hat{f},0)} = \hat{f} \ln\left(\hat{f}/f\right)$ *tal que* $V_{(\hat{f},0)} > 0$ *para todo* $(f, z) \neq (\hat{f}, 0)$ *y* $V_{(\hat{f},0)} = 0$ *para todo* $(f, z) = (\hat{f}, 0)$*. Si*

$$df > \frac{n(1-z)(1+(r-1)z^{n-1}) - r(1-z^n)}{n(1-z)r - r(1-z^n)}, \tag{E.11}$$

entonces $\dot{V}_{(\hat{f},0)} < 0$*, y el punto de equilibrio es localmente asimptóticamente estable.*

En la Figura E.4, se analiza la condición (E.11) para valores de d, con la restricción que $d > d_1$, dado que es un requisito para que el punto de equilibrio sea estable (ver Lema 4). Se puede observar dos regiones en el simplex. La región de color naranja contiene todos los estados del sistema en donde $\dot{V}_{\hat{f}} < 0$, mientras que la región de color púrpura contiene los estados en donde $\dot{V}_{\hat{f}} > 0$. Ambas regiones están separadas por una línea roja que divide el simplex desde un punto en el borde xy (donde $z = 0$) definido como p_1 a otro punto en el borde yz (donde $x = 0$) definido como p_2 (ver Figura E.4). Esta división corresponde al conjunto de puntos en donde $\dot{V}_{\hat{f}}$ cambia de signo.

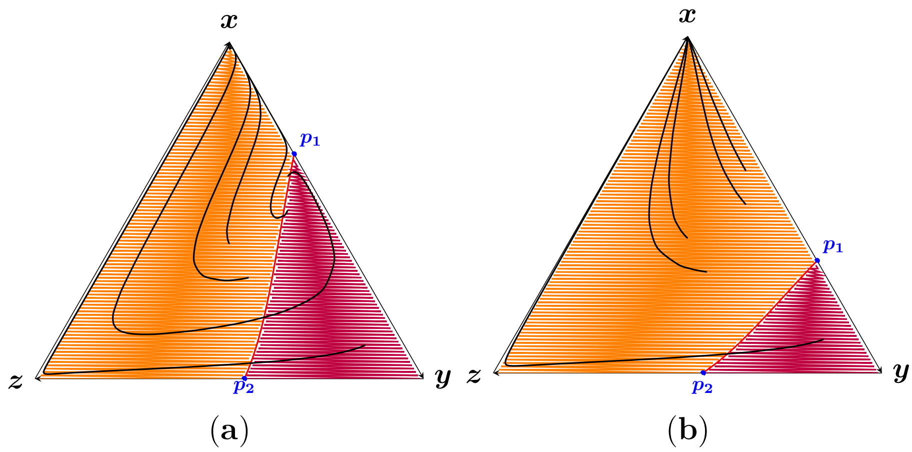


Figura E.4: Valores de $\dot{V}_{\hat{f}}$ para cada estado del sistema. El área en color naranja representa estados donde $\dot{V}_{\hat{f}} < 0$; el área en color púrpura representa estados donde $\dot{V}_{\hat{f}} > 0$. La línea roja entre ambas regiones representa el conjunto de puntos donde $\dot{V}_{\hat{f}}$ cambia de signo. Las líneas en negro, son trayectorias desde diferentes valores iniciales en el simplex. Parámetros: $n = 5$, $r = 3$, $\sigma = 1$ y, d: **(a)** $d = 0.25$ y **(b)** $d = 0.5$.

A medida que el valor de d aumenta, la región donde $\dot{V}_{\hat{f}} < 0$ aumenta de tamaño. Esto ocurre porque el punto p_1 se mueve hacia el vértice $y = 1$. Note que, el valor de p_1 obtenido de $\dot{V}_{\hat{f}} = 0$ es $x = (n - r)/r(n - 1)d$. Esta es la posición del punto de equilibrio interior en el borde xy analizado en el Capítulo D. El punto p_2, en cambio, no depende de d. Cuando $x = 0$, la expresión se reduce a $1 + (r - 1)z^{n-1} - ra = 0$ (ver (B.6)). Por lo tanto, el valor de z en el punto p_2 es fijo y corresponde al valor de $\hat{z}$ cuando $d = 0$ como en (Hauert et al., 2002a).

En la Figura E.4, pueden observarse trayectorias que inician en ambas regiones:$\dot{V}_{\hat{f}} < 0$ y $\dot{V}_{\hat{f}} > 0$. Debido a la estructura de la solución, aún las trayectorias que inician fuera de la vacía de atracción son atraídas y eventualmente terminan en el punto de equilibrio $(1, 0, 0)$ que es, en ese caso, globalmente asimptóticamente estable. Note como en la Figura E.4.(a) algunas trayectorias oscilan antes de alcanzar el equilibrio; esto es causado por el punto de equilibrio inestable que existe en el interior del simplex. Este punto se desplaza hacia el borde xy a medida que d aumenta, y lo alcanza cuando $d = d_2$, que para los parámetros en la Figura E.4, corresponde a $d_2 = 0.2857$.

Desde un punto de vista práctico, son interesantes las trayectorias en las que la frecuencia de cooperadores se incrementa hasta alcanzar el equilibrio. Para observar este comportamiento, se define como Ω al conjunto de valores iniciales donde la frecuencia de cooperadores se incrementa continuamente a lo largo de la trayectoria de la solución hasta alcanzar el punto de equilibrio, i.e.,

$$\Omega := \Big\{(x(t_0), y(t_0), z(t_0)) : x(\tilde{t}) \le x(t)\Big\} \quad \forall t_0 \le \tilde{t} \le t. \tag{E.12}$$

En la Figura 6.7, se observa la región Ω en color verde. A medida que el valor de d aumenta, la región Ω también aumenta. Por otra parte, se puede observar en color azul al conjunto de valores iniciales en el simplex $\mathcal{S}_3$, en los cuales la frecuencia de cooperadores decrece antes de alcanzar el equilibrio, *i.e.* $\mathcal{S}_3 - \Omega$.

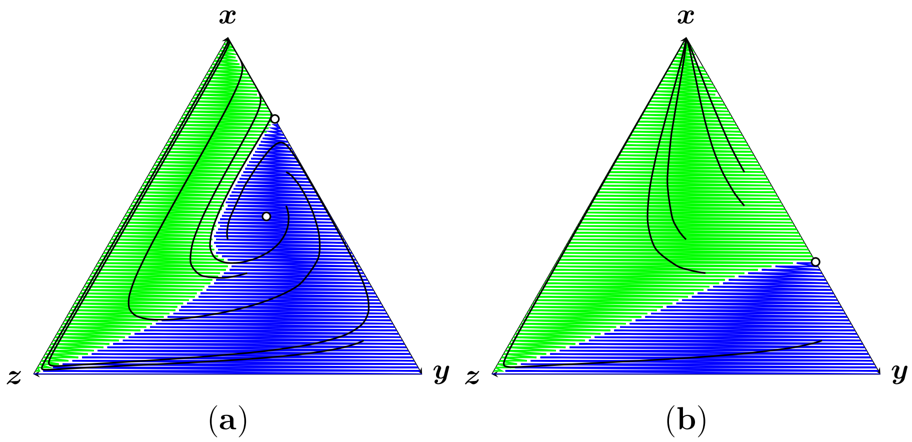


Figura E.5: Valores de x en las trayectorias para cada valor inicial posible. El área verde representa valores iniciales donde la frecuencia de cooperadores x aumenta a lo largo de la trayectoria; el área azul, representa valores iniciales en donde en algún punto de la trayectoria el valor de x disminuye. Las líneas negras corresponden a trayectorias desde diferentes puntos iniciales. Parámetros: $n = 5$, $r = 3$, y $\sigma = 1$, y d (**a**) $d = 0.22$ y (**b**) $d = 0.5$.

E.4 Efecto del castigo fraccionado en el sistema. Resumen.

El efecto del parámetro d (y por lo tanto, del castigo fraccionado) sobre la dinámica del sistema y sobre los puntos de equilibrios del mismo puede describirse utilizando los umbrales d_1 y d_2 introducidos en la sección E.2. (i) Cuando $d = 0$, el punto de equilibrio interior q_1 es un centro, y los vértices son equilibrios tipo silla por lo que el borde del simplex forma una órbita heteroclínica (Hauert et al., 2002a). Las órbitas periódicas alrededor del equilibrio interior pueden verse en la Figure E.6(i). (ii) Si d es positivo, se producen dos transformaciones en el sistema. En primer lugar, el punto de equilibrio q_1 se vuelve inestable y, segundo, la posición del mismo se desplaza hacia el borde xy. En la Figura E.6(ii), se puede ver cómo las trayectorias oscilan, alejándose del punto de equilibrio hacia el borde del simplex.

(iii) Cuando $d_1 < d < d_2$, el punto de equilibrio q_2 surge en el borde xy; es un equilibrio inestable que se desplaza hacia el vértice $y = 1$ a medida que aumenta el valor de d. Además, el equilibrio $x = 1$ que era anteriormente un punto inestable tipo silla, se vuelve un atractor (estable). Asimismo, la cantidad de oscilaciones en las trayectorias, disminuye a medida que d aumenta (ver Figura E.6.(iii) y E.6.(iv)). (v) Cuando $d = d_2$, el equilibrio q_1 alcanza el borde y se une con el punto q_2. Este nuevo punto denominado q_3 continúa siendo inestable (ver Figura E.6.(v)). (vi) Para $d_2 < d$ hasta $d = 1$, no existe un punto de equilibrio en el interior del simplex; el punto q_3 se desplaza hacia el vértice $y = 1$ a medida que d aumenta pero no lo alcanza; el valor de x cuando $d = 1$, es $\hat{x} = (n - r)/r(n - 1)$, que corresponde a la cantidad mínima de cooperadores necesaria para obtener la cooperación en un juego compulsorio, como fuera analizado en el Capítulo D.

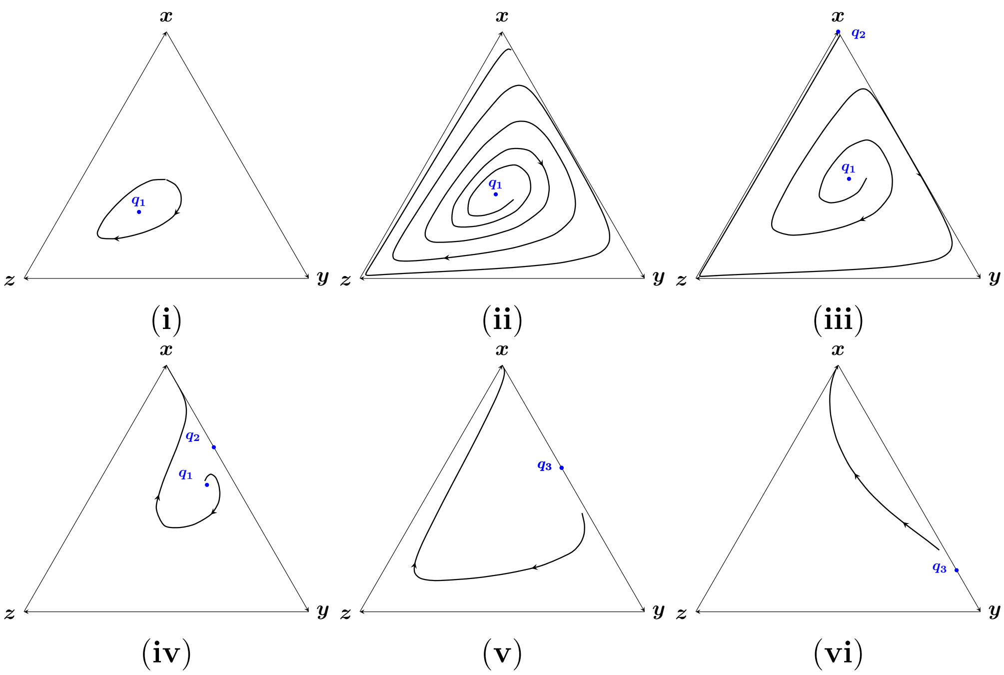


Figura E.6: Diagrama de fases del sistema para diferentes valores de d. Parámetros: $n = 5$, $r = 3$, y $\sigma = 1$. Para estos parámetros: $d_1 = 0.16667$ y $d_2 = 0.2857$. Note que q_1, q_2, y q_3 son puntos de equilibrio. **(i)** $d = 0$, **(ii)** $d = 0.1$, **(iii)** $d = d_1$, **(iv)** $d = 0.25$, **(v)** $d = d_2$, y **(vi)** $d = 0.99$.

Capítulo F

Discusión

El juego de bienes públicos, tanto en la variante clásica compulsoria en donde los individuos deben participar del juego, como en la variante opcional en donde participar es una decisión personal, asume condiciones que no son fácilmente encontradas en la vida real más allá de los experimentos que puedan realizarse en forma controlada en un laboratorio. Si por encima de todo esto, se decide estudiar estos juegos desde el punto de vista evolutivo con una dinámica como, por ejemplo, la dinámica del replicador, la posibilidad de encontrar un caso práctico en donde se cumplan en forma directa todos los requisitos de los modelos, es aún menor. Sin embargo, la problemática que representan estos modelos forma parte el día a día de todos los individuos y, aún cuando no se cumplan los requisitos del modelo teórico, muchos de los resultados observados en los mismos puede relacionarse a situaciones que atraviesan instituciones y proyectos de todo tipo.

Esta duplicidad entre lo teórico y su posible explicación práctica es interesante e intrigante. Varios resultados teóricos tienen sentido en la realidad: (1) como se mencionó anteriormente, aumentar el costo de un servicio porque los recursos son insuficientes, significa castigar a quien está cumpliendo con su obligación y fácilmente puede incentivar a más personas a infringir las reglas, (2) el ciclo característico de la variante del juego opcional, es común en muchos proyectos en donde una etapa exitosa puede desembocar en una situación mala por el incremento de la morosidad hasta el punto que un proyecto puede desaparecer, o (3) como se puede observarse en la Figura E.6 la sanción de una fracción de los infractores disminuye el porcentaje de solitarios en la población en el punto de equilibrio, una consecuencia que parece razonable ya que un proyecto en donde se cumplen las reglas, obtiene mejores resultados y le da mayor confianza a la población para participar en el mismo.

La contribución de esta tesis consiste en un modelo teórico en el que se considera un mecanismo de aplicación de sanciones, ampliamente utilizado en la práctica, que consiste en sancionar solamente a una fracción de los infractores, al que se denominó castigo fraccionado. Para definir el modelo se consideraron dos situaciones comunes relacionadas a la administración de las instituciones: (1) no es posible castigar a todos los infractores por falta de recursos y (2) tanto el servicio proveído como los recursos necesarios para sancionar provienen del monto recaudado por el pago del servicio.

El resultado de aplicar el castigo fraccionado, en el modelo teórico, es el incremento del nivel de cooperación. Inclusive se puede obtener la cooperación de toda la población. El valor de d define también el tipo de trayectoria de la solución, con una d muy pequeña se alcanza la cooperación total después de oscilaciones cada vez mayores, cuando d es mayor las oscilaciones disminuyen. De especial interés son las trayectorias con valores de cooperación siempre mayores hasta alcanzar la cooperación total. Además, el método de aplicar la sanción asegura que no existan desertores de segundo orden, es decir, cooperadores que no aporten al costo de sancionar a los desertores, evitándose el costo de la sanción de segundo orden. Además, el costo total de sancionar depende de la cantidad de desertores a ser sancionados.

Ahora, si se considera una situación similar a la modelada pero en la vida real, y en especial un caso como los proyectos comunitarios en donde los recursos pueden ser escasos, el castigo fraccionado es una herramienta útil para incrementar el nivel de cooperación; además, visto que el costo de la sanción está relacionado con la fracción de individuos a sancionar, la misma puede adecuarse a los recursos disponibles en ese momento en el proyecto. Sin embargo con esta metodología, la presión en el uso correcto de la contribución total es mayor, alcanzar un balance entre el monto a utilizar en la provisión de un servicio y el monto a utilizar para sancionar a los infractores (con el fin de incrementar nuevamente la cooperación y como consecuencia los recursos) es más importante. Se debe tener en cuenta que sancionar a los infractores aumenta la cooperación pero disminuye el beneficio individual. De ahí que alcanzar el balance adecuado entre ambos gastos es quizás una característica más de una institución o proyecto exitoso.

Capítulo G

Conclusión y trabajos futuros

En este trabajo se ha definido el mecanismo de sanción o castigo fraccionado. Se ha modelado el sistema en el contexto de la teoría de juegos evolutivos y la evolución de la cooperación y se ha implementado el castigo fraccionado en un juego de bienes públicos opcional y compulsivo utilizando la dinámica del replicador. Los resultados obtenidos fueron analizados en relación a la modificación de los puntos de equilibrio del sistema y el incremento de la frecuencia de cooperadores y se presentaron los requisitos necesarios para alcanzar la cooperación total. Además, se han señalado las coincidencias entre los resultados teóricos y observaciones empíricas en casos prácticos como el proyecto comunitario de las juntas de saneamiento.

En trabajos futuros, se tendrá en cuenta otros factores que pueden afectar el nivel de cooperación como es la redistribución del beneficio sustraído a los infractores en el grupo de cooperadores así como sanciones parciales. También se considerará una sanción adaptativa que sería más eficiente y que acompañe en el tiempo las modificaciones en la composición del grupo. Además se estudiará la relación entre el porcentaje de la contribución dirigida a sancionar y el nivel de cooperación alcanzado.